ZERO DYNAMICS OF SISO QUASI-POLYNOMIAL SYSTEMS

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Chapter 1

Introduction

Control of nonlinear systems presents a number of challenging nontrivial problems. Appropriate input-output controller structure design for multiple input-multiple output (MIMO) systems is one among them, that may be done using zero dynamics.

Most often, a complex nonlinear system description can cause difficulties in investigating basic system properties such as controllability, observability or stability. For the latter, a well-known and frequently used method, the linearization of the system in a neighborhood of an equilibrium point is usually improper. In most cases, even if the system is locally stable the stability region is so narrow that linear methods cannot be efficiently used for controller design.

Better results can be achieved using quadratic or other simple Lyapunov function candidates for stability region estimation. However, this method has a serious disadvantage: in spite of the existence of the Lyapunov function, the calculation itself is not trivial.

During the decades different special nonlinear system description forms were published to make easier the solution of this problem, such as quasi-polynomial (QP) [6] or Hamiltonian [5] forms. For these system classes there are methods to compute Lyapunov function candidates, which are quadratic, by using linear matrix inequalities (LMIs) [7] or bilinear matrix inequalities (BMIs).

For efficient controller design not only the stability of a desired equilibrium should be investigated but the stability of the zero dynamics around this point itself. Analysis of the zero dynamics has a key role in controller structure selection. When the value of one state variable is selected as an output (a special partial state feedback), then the system can be forced to have constant valued output by using a stable output feedback. If the system has stable zero dynamics then not only that state variable will remain at its equilibrium value but all other state variables will converge asymptotically to their equilibrium values in case of a disturbance. However, e.g. in process systems, unstable zero dynamics can make the remaining state variables of the system go far from the desired values and reach an other equilibrium point. Form this follows that an appropriate input-output pair which possesses stable zero dynamics can be used for efficient controller structure selection.

Chapter 2

Basic notions

Every process system and most nonlinear systems can be written in a so-called input-affine form:

$$\begin{array}{rcl} \dot{x} & = & f(x) + \sum_{i=1}^{m} g_i(x) u_i \quad \textit{(state equation)} \\ y_j & = & h_j(x) \quad\quad j = 1, \dots, p \quad \textit{(output equation)} \end{array} \tag{2.1}$$

where $u_i, \quad i = 1, \dots, m$ denotes the control input, and f, g_i and h are smooth nonlinear functions. The $m = p = 1$ case is called single input-single output (SISO) system. An input-output pair of input-affine systems can be characterized by its *relative degree*. This value is defined at an equilibrium point x^0, and shows the number of times that the output should be differentiated so that the input appeared in its derivative.

The exact definition is the following: given a SISO nonlinear system in input-affine form

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x) u \\ y & = & h(x) \end{array} \tag{2.2}$$

with an x_0 equilibrium point. Let U be a neighborhood of x_0. The system above has relative degree r if

1. $L_g L_f^k h(x) = 0 \quad \forall x \in U$, $k < r - 1$, and
2. $L_g L_f^{r-1} h(x_0) \neq 0$

The latter condition is not necessarily fulfilled so the relative degree is not always defined. If r is less than the number of state variables n then the *zero dynamics* describes the dynamical behavior of the system when its output is forced to be identically zero (generally: constant) by a suitable static nonlinear feedback [3]. The dimension of this dynamics equals to the difference between the number of state variables and the value of the relative degree.

Every lumped process system can be written in *quasi polynomial* (QP) form [6]:

$$\dot{x}_i = x_i(\lambda_i + \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}) \tag{2.3}$$

$$i = 1, \dots, n \quad m \geq n, \quad x_i > 0$$

where the constant vector $\lambda \in \mathbb{R}^n$ and the real valued constant matrices $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times n}$ are the system parameters. The number m is related to the number of *quasi monomials* defined by the terms $z = \prod_{k=1}^{n} x_k^{B_{jk}}$. Choosing these monomials as state variables results in the Lotka–Volterra form [2]:

$$\dot{z}_i = z_i(\lambda_i^{LV} + \sum_{j=1}^{m} M_{ij} z_j) \tag{2.4}$$

$$i = 1, \ldots, m$$

where $\lambda^{LV} = B\lambda \in \mathbb{R}^m$ and $M = BA \in \mathbb{R}^{m \times m}$. The importance of these representations is the easier *stability analysis* of nonlinear systems. However, its is important to note that if the number of quasi monomials is greater than the number of state variables then the matrix M is singular. Singularity causes the appearance of zero eigenvalues with their number equal to $m - n$. These eigenvalues should not be taken into account, but only the remaining ones can be used for local stability analysis.

A technical tool for stability analysis is the usage of *linear matrix inequalities* (LMIs):

$$F(z) = F_0 + \sum_{i=1}^{m} z_i F_i \geq 0 \tag{2.5}$$

where $z \in \mathbb{R}^m$ is the variable and $F_i \in \mathbb{R}^{n \times n}$, $i = 0, \ldots, m$ are given symmetric matrices. The inequality symbol represents the positive semi-definiteness of $F(z)$.

Chapter 3

Zero dynamics of LILO QP systems

The system description in (2.3) defines an autonomous system. However, defining a parameter u as an input the constant vector λ can be decomposed as

$$\lambda_i + k_i u, \quad i = 1, \ldots, n \tag{3.1}$$

This will mean a linear time-invariant relation between the input and the derivative of the state variables. Using the decomposition above, quasi polynomial systems with linear input can be defined:

$$\dot{x}_i = x_i(\lambda_i + \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} + k_i u) \tag{3.2}$$

The above equation implies that $g(x)$ is a linear function.

Extending the state equations with an output equation

$$y = h(x) \tag{3.3}$$

where $h(x)$ is a linear function, e.g. $h_1 x_1$, results a linear input-linear output (LILO) QP system.

If one usees the LILO QP form of a nonlinear system, then explicit formulae can be derived for its zero dynamics. Calculations were made at first for the simpler $r = 1$ and then the $r > 1$ case. The input variable of the system has been fixed and the zero dynamics for different possible output variable candidates was investigated. For the $r = 1$ case stability analysis of the original system and of the zero dynamical description can be performed, as well.

3.1 The $r = 1$ case

The QP system (3.2) can be written into input-affine form (2.1) with:

$$f(x) = \begin{bmatrix} x_1(\lambda_1 + \sum_{j=1}^{m} A_{1j} \prod_{k=1}^{n} x_k^{B_{jk}}) \\ x_2(\lambda_2 + \sum_{j=1}^{m} A_{2j} \prod_{k=1}^{n} x_k^{B_{jk}}) \\ \vdots \\ x_n(\lambda_n + \sum_{j=1}^{m} A_{nj} \prod_{k=1}^{n} x_k^{B_{jk}}) \end{bmatrix}, \quad g(x) = \begin{bmatrix} x_1 k_1 \\ x_2 k_2 \\ \vdots \\ x_n k_n \end{bmatrix}, \quad k_1 \neq 0 \tag{3.4}$$

The output of the QP system $y = h(x) = h_1 x_1$ is selected as a linear function of the state variable x_1. In this case the system has relative degree 1 around an equilibrium x^0. Efficient output feedback type control can be done only when the system has stable zero dynamics (at least locally around an operating point). In order to determine the equations describing the dynamical behavior a special input is selected:

$$u = -\frac{L_f h(x)}{L_g h(x)} = -\frac{1}{k_1}(\lambda_1 + \sum_{j=1}^{m} A_{1j} \prod_{k=1}^{n} x_k^{B_{jk}}) \tag{3.5}$$

This u above is called *output zeroing input* [8] and results in the following zero dynamics:

$$\dot{x} = \begin{bmatrix} 0 \\ x_2(\lambda_2 - \frac{k_2}{k_1}\lambda_1 + \sum_{j=1}^{m}(A_{2j} - \frac{k_2}{k_1}A_{1j})(x_1^0)^B_{j1} \prod_{k=2}^{n} x_k^{B_{jk}}) \\ \vdots \\ x_n(\lambda_n - \frac{k_n}{k_1}\lambda_1 + \sum_{j=1}^{m}(A_{nj} - \frac{k_n}{k_1}A_{1j})(x_1^0)^B_{j1} \prod_{k=2}^{n} x_k^{B_{jk}}) \end{bmatrix} \tag{3.6}$$

It is seen that the zero dynamics is given by an autonomous system in QP-form, the stability of which can be investigated by using e.g. linear matrix inequalities. The state variable x_1 selected as an output is eliminated according to the notion of zero dynamics ($y = h_1 x_1^0 \Rightarrow x_1 = x_1^0$). From this follows that quasi monomials became simpler because the monomials differing in the power of x_1 will unite in one monomial with different coefficients. Different powers of x_1 became constant and unite with λ. An example of this is given later in section 3.4

3.2 The $r > 1$ case

Relative degree r will increase when such an output is selected which is not affected directly by the input, i.e. $k_i = 0,\ i \in \{1, \dots, n\}$. In these cases the output should be differentiated at least twice so that the input appeared in it. Higher r value requires many symbolic calculations which may take a lot of time and be unnecessarily complex. Having determined r, the output zeroing input can be calculated as:

$$u = -\frac{L_f^r h(x)}{L_g L_f^{r-1} h(x)} \tag{3.7}$$

It is important to note, however, that a system description may lose its quasi polynomial form with a feedback like in (3.7). For a proof suppose that $y = h_2 x_2$ and $k_2 = 0$, i.e. $r = 2$! Then the output zeroing input will be the following:

$$u = -\frac{L_f^2 h(x)}{L_g L_f h(x)} \tag{3.8}$$

The numerator equals to

$$L_f^2 h(x) = \frac{\partial L_f h(x)}{\partial x_1} f_1(x) + \frac{\partial L_f h(x)}{\partial x_2} f_2(x) + \dots + \frac{\partial L_f h(x)}{\partial x_n} f_n(x) \tag{3.9}$$

where

$$L_f h(x) = h_2 x_2 (\lambda_2 + \sum_{j=1}^{m} A_{2j} \prod_{k=1}^{n} x_k^{B_{jk}})$$

$$\frac{\partial L_f h(x)}{\partial x_1} f_1(x) = \left(h_2 x_2 \sum_{j=1}^{m} A_{2j}^{(1)} \prod_{k=1}^{n} x_k^{B_{jk}^{(1)}} \right) \left(x_1(\lambda_1 + \sum_{j=1}^{m} A_{1j} \prod_{k=1}^{n} x_k^{B_{jk}}) \right)$$

$$\frac{\partial L_f h(x)}{\partial x_2} f_2(x) = \left(h_2(\lambda_2 + \sum_{j=1}^{m} A_{2j} \prod_{k=1}^{n} x_k^{B_{jk}}) + h_2 x_2 \sum_{j=1}^{m} A_{2j}^{(2)} \prod_{k=1}^{n} x_k^{B_{jk}^{(2)}} \right) \left(x_2(\lambda_2 + \sum_{j=1}^{m} A_{1j} \prod_{k=1}^{n} x_k^{B_{jk}}) \right)$$

$$\vdots \tag{3.10}$$

$$\frac{\partial L_f h(x)}{\partial x_n} = \left(h_2 x_2 \sum_{j=1}^{m} A_{2j}^{(n)} \prod_{k=1}^{n} x_k^{B_{jk}^{(n)}} \right) \left(x_n(\lambda_n + \sum_{j=1}^{m} A_{1j} \prod_{k=1}^{n} x_k^{B_{jk}}) \right)$$

The notations $A_{2j}^{(i)}$ and $B_{jk}^{(i)}$, $i = 1, \ldots, n$ are defined as follows:

$$A_{2j}^{(i)} = \begin{cases} 0 & , B_{ji} = 0 \\ A_{2j} B_{ji} & , B_{ji} \neq 0 \end{cases} \quad , B_{jk}^{(i)} \begin{cases} B_{jk} & , k \neq i \\ B_{jk} - 1 & , k = i \end{cases} \tag{3.11}$$

Because new coefficient and power matrices are defined, new quasi monomials appear and will remain in the equations after feedback.

Considering the denominator of u we obtain

$$L_g L_f h(x) = \left(h_2 x_2 \sum_{j=1}^{m} A_{2j}^{(1)} \prod_{k=1}^{n} x_k^{B_{jk}^{(1)}} \right) x_1 k_1 + \tag{3.12}$$

$$+ \left(h_2 x_2 \sum_{j=1}^{m} A_{2j}^{(3)} \prod_{k=1}^{n} x_k^{B_{jk}^{(3)}} \right) x_3 k_3 + \ldots + \left(h_2 x_2 \sum_{j=1}^{m} A_{2j}^{(n)} \prod_{k=1}^{n} x_k^{B_{jk}^{(n)}} \right) x_n k_n$$

it can be seen that the system will remain in quasi polynomial form only when the denominator contains only one quasi monomial. Otherwise, it will lose this property.

3.3 Stability analysis of LILO QP systems

Stability is a well-known system property. As mentioned earlier, its analysis may suffer from computational difficulties in the case of nonlinear systems. However, there are system classes where the problem of stability analysis can be solved, e.g. [2], [5] or [6]. For quasi polynomial systems a realization independent matrix is used in the analysis. Such a matrix is $M = BA$, see eq. (2.4), which has dimension $m \times m$, where m equals to the number of quasi monomials.

Local stability analysis can be performed the same way as in linear system theory. At first, an equilibrium point x^0 is calculated, and from this, steady state values of the quasi monomials (z^0). Then, the matrix

$$M^0 = diag(z^0) M = \begin{bmatrix} z_1^0 & 0 & \cdots & 0 \\ 0 & z_2^0 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & z_m^0 \end{bmatrix} M \tag{3.13}$$

plays the role of the state matrix of a linear time invariant (LTI) system (the linear equivalent of $f(x)$). Stability analysis can be done with the help of the eigenvalues of M^0. If the real parts of all eigenvalues are less than 0, i.e. $\Re(s_i) < 0, i = 1, \ldots, m$, then the system is asymptotically stable in a neighborhood of x^0.

The same can be performed for the zero dynamics (3.6). In this case the dimension of A_{zd} and B_{zd} is smaller than the original of A and B was. However, some quasi monomials become constant, and with their coefficient they unite with λ resulting λ_{zd}. From this follows that neither general form for A_{zd} and B_{zd} nor general conclusion for the stability of an equilibrium point in the zero dynamics, which was stable before feedback, can be drawn. This property depends on many parameters and from this follows that it will not necessarily remain.

Global stability analysis can be performed, after calculating M^0 and M_{zd}, by the following often used Lyapunov function candidate (see e.q. [4]):

$$V(z) = \sum_{i=1}^{m} c_i (z_i - z_i^0 - z_i^0 \ln(\frac{z_i}{z_i^0})) \tag{3.14}$$

where $c_i \in \mathbb{R}$ for $i = 1, \ldots, m$ and $c_1 > 0$. An equilibrium point is globally stable if and only if

$$\dot{V}(z) = \frac{1}{2}(z_i - z_i^0)^T (M^T C + CM)(z_i - z_i^0) \leq 0 \tag{3.15}$$

where $C = \text{diag}(c_1, \ldots, c_m)$. The equation above holds if and only if the following LMI has a positive definite diagonal solution C:

$$M^T C + CM \leq 0 \tag{3.16}$$

It was shown in [1] and [4] that the global stability of M implies the global stability of the equilibrium point of the original QP-model (3.2) corresponding to z^0.

3.4 A simple example

Consider the following SISO system given in quasi polynomial form:

$$\begin{aligned} \dot{x}_1 &= x_1(1 + x_1 x_2 + u) \\ \dot{x}_2 &= x_2(x_1^2 + x_2 + 2u) \\ y &= h(x) = x_1 \end{aligned} \tag{3.17}$$

The quasi monomials are: $x_1 x_2$, x_1^2, x_2. The system has relative degree $r = 1$ at the equilibrium (x_1^0, x_2^0) because the output is directly affected by the input:

$$\dot{h}(x) = \dot{x}_1 = x_1(1 + x_1 x_2 + u) \Rightarrow r = 1$$

Using the definition of zero dynamics the constant output will be

$$x_1 = x_1^0$$

Then the output zeroing input can be calculated as

$$0 = x_1^0(1 + x_1^0 x_2 + u) \Rightarrow u = -1 - x_1^0 x_2 \tag{3.18}$$

The input above results in the following zero dynamics:

$$\dot{x}_2 = x_2((x_1^0)^2 + x_2 - 2 - 2x_1^0 x_2) \tag{3.19}$$

It can be easily seen that the number of the state equations decreased to one, and the number of new quasi monomials decreased, as well (it is only x_2). The monomials differing in the power of x_1 united into x_2 and a constant, respectively. The reason for this is that $y = x_1 = x_1^0$.

For stability analysis the A, B and $M = BA$ matrices are the following:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{3.20}$$

The system has 3 equilibria: a trivial $(x_1^0,\ x_2^0) = (0,\ 0)$, a semi-trivial $(x_1^0,\ x_2^0) = (0,\ -2)$ and at last $(x_1^0,\ x_2^0) = (0.7709,\ -2.5944)$. The first two should not be taken into account because for these the condition $x_i > 0,\ i = 1, \ldots, n$, see eq. (2.3), does not hold. The remaining one may be used for further calculations. Then, the quasi monomials will have the values

$$\begin{bmatrix} x_1^0 x_2^0 \\ (x_1^0)^2 \\ x_2^0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0.5943 \\ -2.5944 \end{bmatrix} \tag{3.21}$$

Using these, the linear state matrix around x^0 can be calculated as

$$M^0 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.5942 & 0 \\ 0 & 0 & -2.5944 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -2 \\ 1.1886 & 0 & 0 \\ 0 & -2.5944 & -2.5944 \end{bmatrix} \tag{3.22}$$

The system has 3 eigenvalues: $s_1 = 0$, $s_{2,3} = -2.2972 \pm 1.5129\mathbf{i}$. As it was mentioned earlier, the matrix M is singular because the dimension of the realization has increased from 2 to 3. This implies that a 0 eigenvalue has appeared. However, for local stability analysis the 0 eigenvalues, coming from the increase of dimension, should not be taken into account. This means that the system has two locally asymptotically stable poles s_2, s_3, i.e. $Re(s_{2,3}) < 0$, at the equilibrium point $(x_1^0,\ x_2^0) = (0.7709,\ -2.5944)$.

Global stability analysis results that there is no positive definite diagonal solution for the eq. (3.16) which implies that the equilibrium above is not globally stable. A possible cause for this may be the semi-trivial equilibrium point. However, this should not be taken into account.

For the zero dynamics the A_{zd}, B_{zd} and M_{zd} matrices will have the form:

$$A_{zd} = \begin{bmatrix} 1 - 2x_1^0 \end{bmatrix}, \quad B_{zd} = \begin{bmatrix} 1 \end{bmatrix} \tag{3.23}$$

$$M_{zd} = \begin{bmatrix} 1 - 2x_1^0 \end{bmatrix} = \begin{bmatrix} -0.5418 \end{bmatrix} \tag{3.24}$$

It can be calculated that the zero dynamics will have an equilibrium point ($x_2^0 = -2.5944$) which is similar to the equilibrium of the open-loop system. However, the eigenvalue $s = 1.4057$ of the matrix M_{zd}^0 has not remained in the left half part of the complex plane, so the stability property has changed. $Re(s) > 0$ means that the system will not remain stable with a feedback in (3.18). A possible reason for this may be that u has not the form of $-Gx$ but $-Gx + k$, i.e. contains a constant.

Chapter 4

Case study: heat exchanger cells

In this section a cascade model of countercurrent heat exchangers is investigated. Countercurrent means that the direction of the hot side flow in the inner tube and the cold side flow in the outer tube is opposite. At first, the model of a single cell is considered [9]. The behavior of this cell can be described by the following differential equations:

$$\begin{aligned}\frac{dT_{co}(t)}{dt} &= \frac{T_{ci}(t)-T_{co}(t)}{V_c}v_c(t) + \frac{UA}{V_c c_{pc}\rho_c}(T_{ho}(t) - T_{co}(t))\\ \frac{dT_{ho}(t)}{dt} &= \frac{T_{hi}(t)-T_{ho}(t)}{V_h}v_h(t) + \frac{UA}{V_h c_{ph}\rho_h}(T_{co}(t) - T_{ho}(t))\end{aligned} \tag{4.1}$$

where $T_{co}, T_{ci}, T_{ho}, T_{hi}$ denote the temperature of the cold side outlet, inlet and the hot side outlet, inlet flows. V_c and V_h denote the volume of the liquid flowing in the tubes respectively (they are assumed to be constant). The notations $v_c(t)$ and $v_h(t)$ are the cold and hot side inlet (and outlet) flow rates. The constants c_{pc}, c_{ph}, ρ_c and ρ_h are the specific heat and the density of the liquid in both sides respectively. At last, A denotes the heat transfer area of the cell and U is the heat transfer coefficient. To construct a quasi polynomial description centered state variables should be defined:

$$\bar{T}_{co}(t) = T_{co}(t) - T_{ci}^0 \tag{4.2}$$

and

$$\bar{T}_{ho}(t) = T_{ho}(t) - T_{hi}^0 \tag{4.3}$$

This gives the centered state equations of a single countercurrent heat exchanger cell

$$\begin{aligned}\frac{d\bar{T}_{co}(t)}{dt} &= -\frac{\bar{T}_{co}(t)}{V_c}v_c(t) + \frac{UA}{V_c c_{pc}\rho_c}(\bar{T}_{ho}(t) + T_{hi}^0 - \bar{T}_{co}(t) - T_{ci}^0))\\ \frac{d\bar{T}_{ho}(t)}{dt} &= -\frac{\bar{T}_{ho}(t)}{V_h}v_h(t) + \frac{UA}{V_h c_{ph}\rho_h}(\bar{T}_{co}(t) + T_{ci}^0 - \bar{T}_{ho}(t) - T_{hi}^0))\end{aligned} \tag{4.4}$$

which can be easily transformed to a LILO quasi polynomial form:

$$\begin{aligned}\frac{d\bar{T}_{co}(t)}{dt} &= \bar{T}_{co}(t)(-\frac{UA}{V_c c_{pc}\rho_c} + \frac{UA}{V_c c_{pc}\rho_c}(\bar{T}_{ho}(t)\bar{T}_{co}^{-1}(t) + (T_{hi}^0 - T_{ci}^0)\bar{T}_{co}^{-1}(t)) - \frac{1}{V_c}v_c(t))\\ \frac{d\bar{T}_{ho}(t)}{dt} &= \bar{T}_{ho}(t)(-\frac{UA}{V_h c_{ph}\rho_h} + \frac{UA}{V_h c_{ph}\rho_h}(\bar{T}_{co}(t)\bar{T}_{ho}^{-1}(t) + (T_{ci}^0 - T_{hi}^0)\bar{T}_{ho}^{-1}(t)) - \frac{1}{V_h}v_h(t))\end{aligned} \tag{4.5}$$

Potential input variable candidates are $v_c(t)$ and $v_h(t)$ and output variable candidates are the state variables $\bar{T}_{co}$ and $\bar{T}_{ho}$.

By using the model of the single heat exchanger cell the cascade model can be defined as

$$\begin{aligned}\frac{dT_{jc}(t)}{dt} &= \frac{T_{(j+1)c}(t)-T_{jc}(t)}{V_{jc}}v_c(t) + \frac{U_jA_j}{V_{jc}c_{pc}\rho_c}(T_{jh}(t) - T_{jc}(t))\\ \frac{dT_{jh}(t)}{dt} &= \frac{T_{(j-1)h}(t)-T_{jh}(t)}{V_{jh}}v_h(t) + \frac{U_jA_j}{V_{jh}c_{ph}\rho_h}(T_{jc}(t) - T_{jh}(t))\end{aligned} \tag{4.6}$$

where $j = 1, \ldots, n$ and $T_{n+1c} = T_{ci},\ T_{1c} = T_{co},\ T_{0h} = T_{hi},\ T_{nh} = T_{ho}$.

According to the centered model in eq. (4.4) the same can be drawn for eq. (4.6):

$$\begin{aligned}
\frac{d\bar{T}_{jc}(t)}{dt} &= \bar{T}_{jc}(t)\left(-\frac{U_jA_j}{V_{jc}c_{pc}\rho_c} + \frac{U_jA_j}{V_{jc}c_{pc}\rho_c}(\bar{T}_{jh}(t)\bar{T}_{jc}^{-1}(t) + (T^0_{(j-1)h} - T^0_{(j+1)c})\bar{T}_{jc}^{-1}(t)) - \frac{1}{V_{jc}}v_c(t)\right) \\
\frac{d\bar{T}_{jh}(t)}{dt} &= \bar{T}_{jh}(t)\left(-\frac{U_jA_j}{V_{jh}c_{ph}\rho_h} + \frac{U_jA_j}{V_{jh}c_{ph}\rho_h}(\bar{T}_{jc}(t)\bar{T}_{jh}^{-1}(t) + (T^0_{(j+1)c} - T^0_{(j-1)h})\bar{T}_{jh}^{-1}(t)) - \frac{1}{V_{jh}}v_h(t)\right)
\end{aligned} \tag{4.7}$$

In further calculations a model with 3 cells is investigated. After defining the centered state variables for $n = 3$

$$\begin{aligned}
\bar{T}_{co}(t) &= T_{co}(t) - T^0_{2c} \\
\bar{T}_{1h}(t) &= T_{1h}(t) - T^0_{hi} \\
\bar{T}_{2c}(t) &= T_{2c}(t) - T^0_{3c} \\
\bar{T}_{2h}(t) &= T_{2h}(t) - T^0_{1h} \\
\bar{T}_{3c}(t) &= T_{3c}(t) - T^0_{ci} \\
\bar{T}_{ho}(t) &= T_{ho}(t) - T^0_{2h}
\end{aligned} \tag{4.8}$$

where

$$\begin{bmatrix} T^0_{2c} \\ T^0_{hi} \\ T^0_{3c} \\ T^0_{1h} \\ T^0_{ci} \\ T^0_{2h} \end{bmatrix} = \begin{bmatrix} 327.3213^\circ K \\ 343^\circ K \\ 324.6446^\circ K \\ 334.278^\circ K \\ 323^\circ K \\ 328.9189^\circ K \end{bmatrix} \tag{4.9}$$

we get the QP description of a cascade countercurrent heat exchanger model:

$$\begin{aligned}
\frac{d\bar{T}_{co}(t)}{dt} &= \bar{T}_{co}(t)\left(-\frac{U_1A_1}{V_cc_{pc}\rho_c} + \frac{U_1A_1}{V_cc_{pc}\rho_c}(\bar{T}_{1h}(t)\bar{T}_{co}^{-1}(t) + (T^0_{hi} - T^0_{2c})\bar{T}_{co}^{-1}(t)) - \frac{1}{V_c}v_c(t)\right) \\
\frac{d\bar{T}_{1h}(t)}{dt} &= \bar{T}_{1h}(t)\left(-\frac{U_1A_1}{V_hc_{ph}\rho_h} + \frac{U_1A_1}{V_hc_{ph}\rho_h}(\bar{T}_{co}(t)\bar{T}_{1h}^{-1}(t) + (T^0_{2c} - T^0_{hi})\bar{T}_{1h}^{-1}(t)) - \frac{1}{V_h}v_h(t)\right) \\
\frac{d\bar{T}_{2c}(t)}{dt} &= \bar{T}_{2c}(t)\left(-\frac{U_2A_2}{V_cc_{pc}\rho_c} + \frac{U_2A_2}{V_cc_{pc}\rho_c}(\bar{T}_{2h}(t)\bar{T}_{2c}^{-1}(t) + (T^0_{1h} - T^0_{3c})\bar{T}_{2c}^{-1}(t)) - \frac{1}{V_c}v_c(t)\right) \\
\frac{d\bar{T}_{2h}(t)}{dt} &= \bar{T}_{2h}(t)\left(-\frac{U_2A_2}{V_hc_{ph}\rho_h} + \frac{U_2A_2}{V_hc_{ph}\rho_h}(\bar{T}_{2c}(t)\bar{T}_{2h}^{-1}(t) + (T^0_{3c} - T^0_{1h})\bar{T}_{2h}^{-1}(t)) - \frac{1}{V_h}v_h(t)\right) \\
\frac{d\bar{T}_{3c}(t)}{dt} &= \bar{T}_{3c}(t)\left(-\frac{U_3A_3}{V_cc_{pc}\rho_c} + \frac{U_3A_3}{V_cc_{pc}\rho_c}(\bar{T}_{ho}(t)\bar{T}_{3c}^{-1}(t) + (T^0_{2h} - T^0_{ci})\bar{T}_{3c}^{-1}(t)) - \frac{1}{V_c}v_c(t)\right) \\
\frac{d\bar{T}_{ho}(t)}{dt} &= \bar{T}_{ho}(t)\left(-\frac{U_3A_3}{V_hc_{ph}\rho_h} + \frac{U_3A_3}{V_hc_{ph}\rho_h}(\bar{T}_{3c}(t)\bar{T}_{ho}^{-1}(t) + (T^0_{ci} - T^0_{2h})\bar{T}_{ho}^{-1}(t)) - \frac{1}{V_h}v_h(t)\right)
\end{aligned} \tag{4.10}$$

The system description contains 12 quasi monomials:

$$\begin{array}{ll}
\bar{T}_{1h}(t)\bar{T}_{co}^{-1}(t) & \bar{T}_{co}^{-1}(t) \\
\bar{T}_{co}(t)\bar{T}_{1h}^{-1}(t) & \bar{T}_{1h}^{-1}(t) \\
\bar{T}_{2h}(t)\bar{T}_{2c}^{-1}(t) & \bar{T}_{2c}^{-1}(t) \\
\bar{T}_{2c}(t)\bar{T}_{2h}^{-1}(t) & \bar{T}_{2h}^{-1}(t) \\
\bar{T}_{ho}(t)\bar{T}_{3c}^{-1}(t) & \bar{T}_{3c}^{-1}(t) \\
\bar{T}_{3c}(t)\bar{T}_{ho}^{-1}(t) & \bar{T}_{ho}^{-1}(t)
\end{array} \tag{4.11}$$

Using the values in Table 4.1. the model equations will have the form in eq. (4.12) with the following assumptions:

1. The heat transfer coefficients U_1,U_2 and U_3 equal to U,
2. The heat transfer areas A_1,A_2 and A_3 equal to A,
3. The $v_c(t)$, $v_h(t)$ inlet flow rates and $T_{hi}(t)$, $T_{ci}(t)$ temperatures are constant valued.

U	heat transfer coefficient	400	$\frac{W}{m^2K}$
A	heat transfer area	4	m^2
V_c	volume of the cold side liquid	1	m^3
V_h	volume of the hot side liquid	0.5	m^3
c_{pc}	specific heat of the cold side liquid	1910	$\frac{J}{kgK}$
c_{ph}	specific heat of the hot side liquid	1590	$\frac{J}{kgK}$
ρ_c	density of the cold side liquid	1000	$\frac{kg}{m^3}$
ρ_h	density of the hot side liquid	1000	$\frac{kg}{m^3}$
v_c	cold side flow rate	0.0005	$\frac{m^3}{s}$
v_h	hot side flow rate	0.0003	$\frac{m^3}{s}$
T_{ci}	temperature of the cold side inlet flow rate	323	K
T_{hi}	temperature of the hot side inlet flow rate	343	K

Table 4.1: Parameter values and dimensions in the cascade model

$$\begin{aligned}
\frac{d\bar{T}_{co}(t)}{dt} &= \bar{T}_{co}(t)(-0.0013 + 0.0008\bar{T}_{1h}(t)\bar{T}_{co}^{-1}(t) + 0.0131\bar{T}_{co}^{-1}(t)) \\
\frac{d\bar{T}_{1h}(t)}{dt} &= \bar{T}_{1h}(t)(-0.0026 + 0.002\bar{T}_{co}(t)\bar{T}_{1h}^{-1}(t) - 0.0315T_{1h}^{-1}(t)) \\
\frac{d\bar{T}_{2c}(t)}{dt} &= \bar{T}_{2c}(t)(-0.0013 + 0.0008\bar{T}_{2h}(t)\bar{T}_{2c}^{-1}(t) + 0.008\bar{T}_{2c}^{-1}(t)) \\
\frac{d\bar{T}_{2h}(t)}{dt} &= \bar{T}_{2h}(t)(-0.0026 + 0.002\bar{T}_{2c}(t)\bar{T}_{2h}^{-1}(t) - 0.0193\bar{T}_{2h}^{-1}(t)) \\
\frac{d\bar{T}_{3c}(t)}{dt} &= \bar{T}_{3c}(t)(-0.0013 + 0.0008\bar{T}_{ho}(t)\bar{T}_{3c}^{-1}(t) + 0.0049\bar{T}_{3c}^{-1}(t)) \\
\frac{d\bar{T}_{ho}(t)}{dt} &= \bar{T}_{ho}(t)(-0.0026 + 0.002\bar{T}_{3c}(t)\bar{T}_{ho}^{-1}(t) - 0.0119\bar{T}_{ho}^{-1}(t))
\end{aligned} \tag{4.12}$$

The system above has only one equilibrium point, which can be calculated easily, and takes the following values:

$$\begin{bmatrix} \bar{T}_{co} \\ \bar{T}_{1h} \\ \bar{T}_{2c} \\ \bar{T}_{2h} \\ \bar{T}_{3c} \\ \bar{T}_{ho} \end{bmatrix} = \begin{bmatrix} 4.3564°K \\ -8.722°K \\ 2.6767°K \\ -5.359°K \\ 1.6446°K \\ -3.2926°K \end{bmatrix} \tag{4.13}$$

Local stability requires the determination of M which depends on A and B. The matrix $diag(z^0)M$ has 12 eigenvalues from which 6 equal to 0. According to the principles in Section 2. these should not be taken into account. The remaining ones have negative real parts which means that the system is locally asymptotically stable, moreover global, because there is only one equilibrium point in the system.

4.1 The $r = 1$ case

Choosing $v_c(t)$ as an input and $\bar{T}_{co}$ as an output, the system will have relative degree 1 in a neighborhood of $\bar{T}_{co} = \bar{T}_{co}^0$. The output zeroing input can be calculated as

$$v_c(t) = -\frac{U_1 A_1}{c_{pc}\rho_c} + \frac{U_1 A_1}{c_{pc}\rho_c}\left(\frac{1}{\bar{T}_{co}^0}\bar{T}_{1h}(t) + \frac{T_{hi}^0 - T_{2c}^0}{\bar{T}_{co}^0}\right) \tag{4.14}$$

This gives the following 5 dimensional zero dynamics

$$
\begin{array}{rcl}
\frac{d\bar{T}_{co}(t)}{dt} & = & 0 \\
\frac{d\bar{T}_{1h}(t)}{dt} & = & \bar{T}_{1h}(t)(-\frac{U_1A_1}{V_h c_{ph}\rho_h} + \frac{U_1A_1}{V_h c_{pc}\rho_c}(\bar{T}^0_{co} + \bar{T}^0_{2c} - T^0_{hi})\bar{T}^{-1}_{1h}(t) - \frac{1}{V_h}v_h(t)) \\
\frac{d\bar{T}_{2c}(t)}{dt} & = & \bar{T}_{2c}(t)(-\frac{U_2A_2}{V_c c_{pc}\rho_c} + \frac{U_1A_1}{V_c c_{pc}\rho_c}(1 - \frac{T^0_{hi}-T^0_{2c}}{\bar{T}^0_{co}})+ \\
 & & +\frac{U_2A_2}{V_c c_{pc}\rho_c}(\bar{T}_{2h}(t)\bar{T}^{-1}_{2c}(t) + (T^0_{1h} - T^0_{3c})\bar{T}^{-1}_{2c}(t)) - \frac{U_1A_1}{V_c c_{pc}\rho_c \bar{T}^0_{co}}\bar{T}_{1h}(t)) \\
\frac{d\bar{T}_{2h}(t)}{dt} & = & \bar{T}_{2h}(t)(-\frac{U_2A_2}{V_h c_{ph}\rho_h} + \frac{U_2A_2}{V_h c_{ph}\rho_h}(\bar{T}_{2c}(t)\bar{T}^{-1}_{2h}(t) + (T^0_{3c} - T^0_{1h})\bar{T}^{-1}_{2h}(t)) - \frac{1}{V_h}v_h(t)) \\
\frac{d\bar{T}_{3c}(t)}{dt} & = & \bar{T}_{3c}(t)(-\frac{U_3A_3}{V_c c_{pc}\rho_c} + \frac{U_1A_1}{V_c c_{pc}\rho_c}(1 - \frac{T^0_{hi}-T^0_{2c}}{\bar{T}^0_{co}})+ \\
 & & +\frac{U_3A_3}{V_c c_{pc}\rho_c}(\bar{T}_{ho}(t)\bar{T}^{-1}_{3c}(t) + (T^0_{2h} - T^0_{ci})\bar{T}^{-1}_{3c}(t)) - \frac{U_1A_1}{V_c c_{pc}\rho_c \bar{T}^0_{co}}\bar{T}_{1h}(t))) \\
\frac{d\bar{T}_{ho}(t)}{dt} & = & \bar{T}_{ho}(t)(-\frac{U_3A_3}{V_h c_{ph}\rho_h} + \frac{U_3A_3}{V_h c_{ph}\rho_h}(\bar{T}_{3c}(t)\bar{T}^{-1}_{ho}(t) + (T^0_{ci} - T^0_{2h})\bar{T}^{-1}_{ho}(t)) - \frac{1}{V_h}v_h(t))
\end{array}
\quad (4.15)
$$

which will have the following form after replacing the parameter values:

$$
\begin{array}{rcl}
\frac{d\bar{T}_{co}(t)}{dt} & = & 0 \\
\frac{d\bar{T}_{1h}(t)}{dt} & = & \bar{T}_{1h}(t)(-0.0026 - 0.0227\bar{T}^{-1}_{1h}(t)) \\
\frac{d\bar{T}_{2c}(t)}{dt} & = & \bar{T}_{2c}(t)(-0.0035 + 0.0008\bar{T}_{2h}(t)\bar{T}^{-1}_{2c}(t) + 0.008\bar{T}^{-1}_{2c}(t) - 0.0002\bar{T}_{1h}) \\
\frac{d\bar{T}_{2h}(t)}{dt} & = & \bar{T}_{2h}(t)(-0.0026 + 0.002\bar{T}_{2c}(t)\bar{T}^{-1}_{2h}(t) - 0.0193\bar{T}^{-1}_{2h}(t)) \\
\frac{d\bar{T}_{3c}(t)}{dt} & = & \bar{T}_{3c}(t)(-0.0035 + 0.0008\bar{T}_{ho}(t)\bar{T}^{-1}_{3c}(t) + 0.0049\bar{T}^{-1}_{3c}(t) - 0.0002\bar{T}_{1h}) \\
\frac{d\bar{T}_{ho}(t)}{dt} & = & \bar{T}_{ho}(t)(-0.0026 + 0.002\bar{T}_{3c}(t)\bar{T}^{-1}_{ho}(t) - 0.0119\bar{T}^{-1}_{ho}(t))
\end{array}
\quad (4.16)
$$

As a result of using a feedback (4.14) the number of quasi monomials decreased by 2. The following monomials disappeared:

$$\bar{T}_{1h}(t)\bar{T}^{-1}_{co}(t), \quad \bar{T}^{-1}_{co}(t), \quad \bar{T}_{co}(t)\bar{T}^{-1}_{1h}(t) \quad (4.17)$$

and a new quasi monomial was created from $\bar{T}_{1h}(t)\bar{T}^{-1}_{co}(t)$:

$$\bar{T}_{1h}(t) \quad (4.18)$$

It can be seen that the system description became simpler: the number of equations decreased by 1 and some quasi monomials are eliminated because the state variable $\bar{T}_{co}$ became constant. The $v_c(t) - \bar{T}_{co}(t)$ input-output pair for controller structure selection is preferred.

From the differential equations describing the zero dynamics of the cascade model, the coefficient matrix A_{zd} and power matrix B_{zd} can be easily derived. Then, for local and global stability analysis the matrix M_{zd} can be calculated as $M_{zd} = B_{zd}A_{zd}$. If the positive definite diagonal solution C of the following equation exists

$$M^T_{zd}C + CM_{zd} \leq 0 \quad (4.19)$$

then the matrix M_{zd} is diagonally stabilizable, i.e. the equilibrium formed by the remaining 10 quasi monomials are globally stable.

Fortunately, in the case of the heat exchanger local stability analysis gives global results because the zero dynamics have only one equilibrium point as the original description has. Calculations showed that this point is globally asymptotically stable, i.e. there is no need to solve the LMI in (4.19).

4.2 The $r > 1$ case

It is an important question if the system remains in quasi polynomial form with feedback like in (3.7). The answer lies in the denominator of u. If it contains more than one quasi monomial then the closed-loop system will lose its QP form.

When $v_c(t)$ is the input of the system and $\bar{T}_{ho}$ is selected as in output then the relative degree of the system will be 2. In this case it is easy to see that the condition above does not hold so the system will not remain in quasi polynomial form.

Chapter 5

Conclusions

Quasi polynomial (QP) form can help in investigating key dynamical properties, such as zero dynamics of nonlinear systems.

For QP systems with relative degree $r = 1$ a simple closed formulae is proposed to calculate its zero dynamics. It is shown that the zero dynamics is in QP form and the description is simpler (with fewer quasi monomials). Thus the input-output pairs with $r = 1$ are always preferable for controller structure selections. However, in most cases when $r > 1$ the QP property may disappear when applying output zeroing feedback.

A countercurrent heat exchanger with three cells was investigated for a fixed input and different output selections. In the $r = 1$ case the closed formulae was used to determine the zero dynamics of the system. The QP form became simpler and the number of quasi monomials decreased. For $r > 1$ the condition of remaining in QP form did not hold.

Stability analysis showed for the $r = 1$ case that not even the original system was globally asymptotically stable around the equilibrium but the zero dynamics.

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