Towards optimal quantum state estimation of a qubit by using indirect measurements

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Technical report SCL-002/2008

Abstract

The properties of the indirect measurement scheme is investigated in this report in the *discrete time case*. The simplest possible case is considered, where both the unknown and the measurement quantum systems are quantum bits. The measurements applied on the measurement qubit are the classical von Neumann measurements using the Pauli matrices as observables.
The statistical properties of the estimate in terms of the variance of the ML estimator and the non-demolition probability are analytically calculated in a simple case when the measurements are applied in the x direction while the qubits interact in the y direction and the initial state of the measurement qubit is $[0, 0, c]$. A way of finding an optimal compromising measurement strategy between the asymptotic variance and the non-demolition probability is also proposed.
The efficiency of the results have been compared with a classical 'standard' state estimation procedure available in the literature. Although the classical one performs better by means of the variance, the indirect one gives a degree of freedom in the above mentioned trade-off problem. The estimation method has also been modified in a few ways to improve its precision.

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Chapter 1

Introduction

State estimation is a fundamental problem in both quantum information theory and quantum control. In quantum control [1] it's role is the same as in the classical control theory, i.e. to give an estimate of the unmeasured time-dependent state variable in order to be used in state feedback schemes. On the other hand, the measurement of a quantum mechanical system is probabilistic, so even the measurement of a measurable quantity asks for estimation methods, this is why state estimation is an important field in quantum information theory [9, 10] , too.

To set up a quantum state estimation, or a *quantum state tomography* method, two ingredients has to be given: the *measurement strategy* used for getting information, and the *estimator* mapping the measurement data to the state space. If one uses a von Neumann measurement, the projective nature of the measurement forces the use of several copies of the same system being in the same state [12] . Methods using this kind of measurement scheme are useless, if one aims at the estimation of dynamically evolving states, as in quantum control, since in general, the many identical copies of the system is impossible to implement in practice. However, in certain physical circumstances, for example in quantum optics, it is natural to have several copies of the state.

A possible way to circumvent the obstruction of the demolition property of von Neumann measurements is to use an *indirect measurement scheme*, where the 'unknown' quantum system is coupled with a 'measurement' (also called 'probe' or 'ancilla') system and the measurements are only applied on the measurement system [6] . In the literature this method is often termed *weak measurement* [14, 3] . Note, that most of the papers dealing with indirect or weak measurement schemes use a continuous-time approach [5] .

However, it is intuitively clear, that one must make a compromise between the information gained in a measurement and the disturbance or demolition caused by it. The general impossibility of determining the state of a single quantum system is proved in [2] whatever measurement scheme is used. This indicates that the efficiency or precision provided by an indirect measurement scheme is necessarily smaller than that of a scheme that uses von Neumann measurements.

A related problem to the state estimation is to prepare the state of a given system in a specified way. Most papers apply some kind special dedicated measurements either to drive the system into a desired state or to compensate for the 'measurement back-action'. An application of weak measurements in bipartite state purification can be seen in [4] , where the authors also use continuous time dynamics. Korotkov and Jordan [8] have shown that "it is possible to fully restore any unknown, pre-measured state, though with probability less than unity."

The aim of this paper is to investigate the properties of the indirect measurement scheme in the *discrete time case*. The simplest possible case is considered, where both the unknown and the measurement quantum systems are quantum bits. The measurements applied on the measurement qubit are the classical von Neumann measurements using the Pauli matrices as observables.

A further aim is to construct a compromising estimator, that finds the trade-off between the effectiveness of the estimate and the number of qubits that are un-affected by the measurements.

The paper is organized as follows. Chapter 2 clarifies the notation used throughout the work. The scheme of indirect measurement is discussed in Chapter 3 together with the dynamics of the two-qubit system under weak measurement. Section 4 presents a simple indirect measurement strategy, the properties of which is investigated in Chapter 5. Chapter 6 compares the simple strategy to other known strategies and suggests an optimal version, and finally Chapter 7 concludes.

Chapter 2

Basic notions

Some basic notations regarding the representation of quantum states used in the sequel are presented in this section.

2.1 Bloch representation

Throughout the paper the *Bloch-vector* representation of the states of quantum bits is used, i.e.

$$\rho = \frac{1}{2}\begin{bmatrix} 1+\theta_3 & \theta_1 - i\theta_2 \\ \theta_1 + i\theta_2 & 1-\theta_3 \end{bmatrix} = \frac{1}{2}(I + \theta_1\sigma_1 + \theta_2\sigma_2 + \theta_3\sigma_3), \tag{2.1}$$

where σ_i stands for the Pauli-matrices, $i = 1, 2, 3$, and $\theta = [\theta_1, \theta_2, \theta_3]^T$ is the Bloch-vector. This way, the states of a quantum bit can be described with three dimensional real vectors of maximal length 1, i.e. $\theta \in \mathbb{R}^3, \quad ||\theta|| \leq 1$. Thus, the state space of the system is the unit ball in $\mathbb{R}^3$.

2.2 Dynamics of a single qubit

The Schrödinger picture is used here in discrete time that associates a unitary U to the time-evolution of the system such that

$$\rho(k) = U\rho(k-1)U^*, \tag{2.2}$$

where $\rho(k)$ is the density matrix of the system at the time instance k, and

$$U = \exp(-ihH(u_x, u_y, u_z)) \tag{2.3}$$

with $H(u_x, u_y, u_z) = u_x\sigma_1 + u_y\sigma_2 + u_z\sigma_3$ being the Hamiltonian operator of the system, h is the sampling time, and u_x, u_y, and u_z are the inputs.

Instead of the unitary description (2.2) of the dynamics, we will use the so called *T-representation* of the linear mapping $\theta(k-1) \mapsto \theta(k)$ that corresponds to the original state transformation $\rho(k-1) \mapsto \rho(k)$ in (2.2) with a real 3×3 rotation matrix T, such that

$$\theta(k+1) = T(u_x, u_y, u_z)\theta(k), \tag{2.4}$$

where the inputs u_x, u_y, and u_z are in the argument of trigonometrical functions, i.e. can effect only the rotational speed of the state vector but not its length. This equation will be generalized to the case of more than one qubit in order to be the basis of the discrete time state equation of the system.

2.3 von Neumann measurement

The most generally used von Neumann measurement is the measurement of the Pauli operators σ_1, σ_2, or σ_3. If one considers the measurement of the observable σ_1, then the possible outcomes are the different eigenvalues of the observable, i.e. ± 1. The probabilities of the different outcomes are

$$\begin{aligned} \text{Prob}(+1) &= \text{Tr}\rho E_{+1} = \tfrac{1}{2}(1+\theta_1) \\ \text{Prob}(-1) &= \text{Tr}\rho E_{-1} = \tfrac{1}{2}(1-\theta_1) \end{aligned}$$

respectively, where the spectral decomposition of σ_1 is $\sigma_1 = E_{+1} - E_{-1}$.

To represent the above measurement as a *stochastic disturbance* it is important to know what the eigenstates of the measurement are. Measuring σ_1, the state after measurement can be

$$\rho^{\pm 1} = \frac{E_{\pm 1}\rho E_{\pm 1}}{\text{Tr}E_{\pm 1}\rho E_{\pm 1}},$$

depending on the actual outcome. In the Bloch vector representation, these states are

$$\theta^{+1} = \begin{bmatrix} +1, & 0, & 0 \end{bmatrix}^T, \quad \theta^{-1} = \begin{bmatrix} -1, & 0, & 0 \end{bmatrix}^T.$$

Chapter 3

Indirect measurement applied to qubits

In order to compute the effect of an indirect measurement on the coupled unknown-measurement qubit system, it is necessary to write up the dynamics of coupled qubit pairs in their Bloch-vector representation.

In what follows, the 'unknown' qubit is denoted by the subscript $_S$ and the 'measurement' qubit or the *probe system* is denoted by $_M$.

3.1 Dynamics of coupled qubit pairs

Let us denote the Bloch representation of the unknown system and the probe (measurement device) as

$$\rho_S(k) = \frac{1}{2}(I + \theta_S(k)\sigma^S) \quad , \quad \rho_M(k) = \frac{1}{2}(I + \theta_M(k)\sigma^M), \tag{3.1}$$

where θ_S and θ_M are 3 dimensional real vectors, σ^S and σ^M are symbolic vectors constructed from the Pauli operators acting on the Hilbert spaces $\mathcal{H}^S$ and $\mathcal{H}^M$.

The state of the composite system is represented as a 4×4 density matrix $\rho_{S+M}(k)$. The state of the composite system after the interaction is given by

$$\rho_{S+M}(k+1) = U_{S+M}\rho_{S+M}(k)U^*_{S+M} \tag{3.2}$$

where U_{S+M} is the overall system evolution unitary. Since we are interested in the dynamical change of the system S, the first reduced density matrix should only be considered:

$$\rho_S(k+1) = \mathrm{Tr}_1\rho_{S+M}(k+1) = \mathrm{Tr}_M\rho_{S+M}(k+1). \tag{3.3}$$

In order to have a simple parametrization of the interaction (coupling) between the unknown and measurement qubit, the Cartan decomposition [7, 13] of the discrete time evolution unitary U_{S+M} is used in the form

$$U_{S+M} = L_1 e^{ah} L_2 \tag{3.4}$$

where L_1 and L_2 are in $SU(2) \otimes SU(2)$ and $a \in \mathbf{a}$ with

$$\mathbf{a} = \mathrm{i}\ span\{\sigma_1^S \otimes \sigma_1^M, \sigma_2^S \otimes \sigma_2^M, \sigma_3^S \otimes \sigma_3^M\} \tag{3.5}$$

Because both L_1 and L_2 are in a product form, they describe the product of the local dynamical effects L_i^S and L_i^M $(i = 1, 2)$, and the interaction is parameterized by three real parameters a_1, a_2 and a_3.

Therefore, the dynamical equation of qubit S in (3.3) becomes

$$\rho_S(k+1) = L_1^S \mathrm{Tr}_M \left(e^{ah} \tilde{\rho}_S(k) \otimes \tilde{\rho}_M(k) e^{a^* h} \right) L_1^{S*} \tag{3.6}$$

where $L_1 = L_1^S \otimes L_1^M$, $L_2 = L_2^S \otimes L_2^M$ both time dependent, and $\tilde{\rho}_S = L_2^S \rho_S L_2^{S*}$, $\tilde{\rho}_M = L_2^M \rho_M L_2^{M*}$. In order to simplify the forthcoming computations, we consider the case *with no local dynamics*, when $L_i^S = L_i^M = I$ $(i = 1, 2)$.

A simple example is a case when

$$U_{S+M} = e^{-ih(a_1 \sigma_1^S \otimes \sigma_1^M)} \tag{3.7}$$

i.e. the qubits are interacting only in the x direction. Computing the dynamics of the system in Bloch representation we obtain

$$\theta_S(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2a_1 h) & -\sin(2a_1 h)\theta_{M1} \\ 0 & \sin(2a_1 h)\theta_{M1} & \cos(2a_1 h) \end{bmatrix} \theta_S(k) = T(a, h, \theta_M)\theta_S(k) \tag{3.8}$$

if there were no measurements performed.

3.2 Measurement strategy

Indirect measurement means that the projective measurements are performed on the probe system (being in state θ_M) attached to the one we are interested in (θ_S). In the composite system (in state ρ_{S+M}) an indirect measurement corresponds to the observables of the form $I \otimes A_M$, where A_M is a self adjoint operator on the Hilbert space of system M. For the sake of simplicity, it is assumed, that A_M is a Pauli spin operator. The *measurement strategy* is

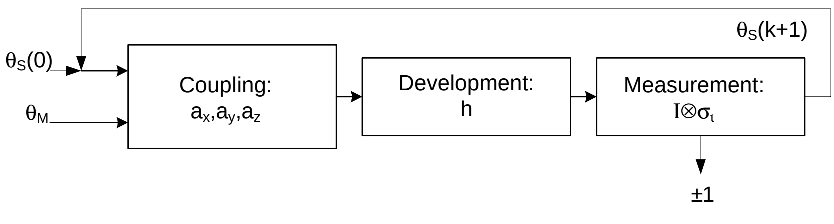

Figure 3.1: Signal flow diagram of indirect measurement

shown in Fig. 3.1. At each time instant of the discrete time set, the measurement qubit is coupled to the unknown system S. They evolve according to the bipartite dynamics (3.6) for the sampling time h, and at the end of the sampling interval, a von Neumann measurement is performed on the measurement qubit. At the next time instant, the previous steps are repeated.

3.3 Parameters of the strategy

The above setting of the indirect measurement allows us to adjust various parameters of the measurement strategy. These can be and will be used to make an optimal compromise between the information gained from the measurement and the demolition caused by the measurement back-action.

The coupling parameters a_1, a_2, a_3 of the Cartan decomposition (3.4-3.5) determine how (in terms of strength and direction) the measurement system is coupled to the unknown one. The sampling time h amplifies this effect and appears as a multiplicative factor to the coupling parameters.

The state of the measurement qubit (θ_M) can be different at each time instant which allows us in the future to introduce a feedback to the measurement protocol.

It is important to note that one can make a 'no information - no demolition' situation by setting the coupling parameters to zero, and a 'maximal information - complete demolition' situation, too. Examples of such extreme cases will be given in the subsection 4.2 and 5.2.

Chapter 4

A simple example for indirect measurement

A simple special case of an indirect measurement is investigated analytically here to brighten the effect of the protocol parameters. As we shall see later, this case can be used to selectively estimate one of the components of the unknown qubit's Bloch vector, similarly to the so called standard measurement scheme [11] for single qubits. A straightforward modification of the measurement setup leads to the estimators of the other two Bloch vector components.

4.1 Measurement setup

Suppose in the sequel that the qubits are interacting only in the y direction for time h (sampling time). Afterwards, an indirect measurement is performed, i.e. a von Neumann measurement of the observable $I \otimes \sigma_x$ on the composite system

$$e^{-\mathrm{i}h(a_2\sigma_2^S \otimes \sigma_2^M)} \cdot (\rho_S \otimes \rho_M) \cdot e^{-\mathrm{i}h(a_2\sigma_2^S \otimes \sigma_2^M)^*},$$

For the sake of simplicity, choose h and a_2 in such a way, that $2a_2h = 2\pi$. The above setting corresponds to the parameters

$$h = \frac{1}{10}, \quad a_2 = \frac{5}{2}\pi, \quad a_1 = a_3 = 0.$$

The probabilities of the different outcomes of $I \otimes \sigma_x$'s measurement are

$$\begin{aligned} \mathrm{Prob}(+1) &= \tfrac{1}{2}(1 + \theta_{S2}\theta_{M3}) \\ \mathrm{Prob}(-1) &= \tfrac{1}{2}(1 - \theta_{S2}\theta_{M3}). \end{aligned} \tag{4.1}$$

Now the probabilities depend on both the state of the unknown θ_S and that of the measurement qubit θ_M. The post-measurement states are

$$\theta_S^{+1} = \begin{bmatrix} \frac{\theta_{S3}\theta_{M2}+\theta_{S1}\theta_{M1}}{1+\theta_{S2}\theta_{M3}} \\ \\ \frac{\theta_{S2}+\theta_{M3}}{1+\theta_{S2}\theta_{M3}} \\ \\ \frac{\theta_{S3}\theta_{M1}-\theta_{S1}\theta_{M2}}{1+\theta_{S2}\theta_{M3}} \end{bmatrix}, \quad \theta_S^{-1} = \begin{bmatrix} \frac{\theta_{S3}\theta_{M2}-\theta_{S1}\theta_{M1}}{1-\theta_{S2}\theta_{M3}} \\ \\ \frac{\theta_{S2}-\theta_{M3}}{1-\theta_{S2}\theta_{M3}} \\ \\ \frac{-\theta_{S3}\theta_{M1}-\theta_{S1}\theta_{M2}}{1-\theta_{S2}\theta_{M3}} \end{bmatrix} \tag{4.2}$$

if $+1$ or -1 was the result, respectively. This measurement setup is useful for unknown state estimation since the probabilities and the new states depend on both θ_S and θ_M. This means, that we both gain information from the measurements and retrieve information in the new states after the measurement.

4.2 Properties of repeated indirect measurements

Let us concentrate on the estimate of the second unknown state co-ordinate, i.e. we want to describe the change of θ_{S2} (notation: $x = \theta_{S2}$) during the measurements. Let us further assume that θ_{M3} is constant (denoted by c) and $\theta_{M1} = \theta_{M2} = 0$, i.e. $\theta_M = [0, 0, c]^T$.

Remark 1 *If $c = 1$ then we get the standard measurement scheme, where*

$$Prob(\pm 1) = \frac{1}{2}(1 \pm x) \quad , \quad x^{\pm 1} = \pm 1$$

It is easy to see from (4.2) that this would be a totally invasive measurement, i.e. the information about the true state would be lost, thus we assume $|c| < 1$.

Proposition 1 *If we measure first +1, and thereafter -1 (or vice versa), then the state of θ_{S2} (=x) will not change.*

Proof: After the first measurement, x changes to $x^{+1} = \frac{x+c}{1+cx}$ then from x^{+1} it turns to be $x^{+1,-1} = \frac{x^{+1}-c}{1-cx^{+1}} = \frac{\frac{x+c}{1+cx}-c}{1-c\frac{x+c}{1+cx}} = \cdots = x$. The reverse goes similarly.

Corollary 1 *All of the possible cases of states can be ordered in a line such a way, that after each measurement we jump in the neighboring state on the left or right side.*

Proposition 2 *If we measure first +1, and thereafter -1 (or vice versa), then the probability of these outcomes doesn't depend on x.*

Proof: First from x it will be $x^{+1} = \frac{x+c}{1+cx}$ with probability: $P = \frac{1}{2}(1 + cx)$ then from x^{+1} it will be x with probability: $Q = \frac{1}{2}(1 - cx^{+1}) = \frac{1}{2}(1 - c\frac{x+c}{1+cx}) = \frac{1}{2}\frac{1-c^2}{1+cx}$. So the probability of this outcome is $P \cdot Q = \frac{1-c^2}{4}$. The reverse goes similarly.

Corollary 2 *If two outcome sequences contain the same number of $+1$ and -1 measurement outcomes, then their probabilities are the same.*

Let us introduce the following notation:

p_n the probability that from n measurements all outcomes are +1s
x_n the resulting state from n measurement when all outcomes are +1s

Corollary 3 *With this notation, the probability that there are k times $+1$ and l times -1 outcomes ($k > l$) in the sequence can be computed as:*

$$\left(\frac{1-c^2}{4}\right)^l \cdot p_{k-l}$$

The state after this sequence of outcomes will be x_{k-l}, and we can represent the sequence of the measurement outcomes as a Markov-process.

Proposition 3 *p_k is a linear function of x.*

Proof: The proof goes by induction. Let $p_k := \frac{q_k}{2^k}$ and $x_k := \frac{y_k}{q_k}$, where (we will prove) q_k and y_k are simple polynomials of c and x.

If $k = 0$ then $q_0 = p_0 = 1$ and $y_0 = x$.

Next let us suppose that both q_k and y_k are simple polynomials, and p_k is linear in x. Then

$$p_{k+1} = p_k \cdot \frac{1}{2}(1 + cx_k) = \frac{q_k}{2^k} \cdot \frac{1}{2}\left(1 + c\frac{y_k}{q_k}\right) = \frac{1}{2^{k+1}} \cdot (q_k + cy_k).$$

On the other hand $p_{k+1} = \frac{q_{k+1}}{2^{k+1}}$, so

$$q_{k+1} = q_k + cy_k. \tag{4.3}$$

Furthermore,

$$x_{k+1} = \frac{x_k + c}{1 + cx_k} = \frac{\frac{y_k}{q_k} + c}{1 + c\frac{y_k}{q_k}} = \frac{y_k + cq_k}{q_k + cy_k} = \frac{y_k + cq_k}{q_{k+1}}.$$

On the other hand $x_{k+1} = \frac{y_{k+1}}{q_{k+1}}$, therefore

$$y_{k+1} = y_k + cq_k. \tag{4.4}$$

Finally we conclude that q_k and y_k are really simple polynomials, and from the recursion it can be seen that both q_k and y_k are linear in x, so p_k is linear in x, too.

Remark 2 *The proof gives us a recursive calculation for x_k and p_k, so it is possible to build up a stochastic model based on the above 3 propositions, and develop a state estimation strategy.*

Chapter 5

Towards optimal quantum state estimation by indirect measurements

Let us suppose that we have N identical copies of the composite quantum system (the two coupled qubits, S and M). We shall use the following *measurement strategy*:

1. Perform 2 subsequent measurements (a *measurement pair*) on each copy with a pre-specified $c = \theta_{M3}$ and compute the maximum-likelihood (ML) estimate of x.
2. Retain the copies on which the measured outcomes were $+1$ and -1 (in any order) for further studies, because they are not affected by the measurements, i.e. their $\theta_{S2} = x$ is left unchanged (see Proposition 1).

Note that the above implies $n = 2$ for the results in sub-section 4.2. Now we investigate how the selection of c (the initial state of the measurement system) affects the variance of the estimate (we want it to be small), and the ratio of the un-affected system copies (we want this to be large).

Denote the number of the $(+1, +1)$ outcomes by N_+, and the probability that a measurement pair result in this outcome by $p_+ = p_2 = \frac{1+c^2+2cx}{4}$. Similarly, the number of the $(-1, -1)$ outcome is denoted by N_-, and its probability is by $p_- = \frac{1+c^2-2cx}{4}$. Then the number of the non-effective ($(+1, -1)$ or $(-1, +1)$) outcomes is $N_0 = N - N_+ - N_-$, and its probability is $p_0 = \frac{1-c^2}{2}$. Then the likelihood function of N measurement pairs is the following polynomial distribution:

$$P = \frac{N!}{N_+!\ N_-!\ N_0!}\ p_+^{N_+}\ p_-^{N_-}\ p_0^{N_0} \tag{5.1}$$

The maximum likelihood estimate of x is obtained by taking the logarithm of P in (5.1), and maximizing it with respect to x:

$$\hat{x}_{ML}(N_+, N_-, c) = \frac{1+c^2}{2c}\frac{N_+ - N_-}{N_+ + N_-} \tag{5.2}$$

This estimate is well-defined if at least one of N_+ or N_- is positive, that holds with probability one when number of measurements goes to infinity. On the other hand, this estimate is asymptotically unbiased.

5.1 The variance and the non-demolition probability

In the case of the investigated measurement setup (see section 4.1), the variance $\mathbf{V}_N$ of the Maximum Likelihood estimator (5.2) is as follows:

$$\begin{aligned} \mathbf{V}_N(c,x) &= \sum_{i=1}^{N} Var\left(\tfrac{1+c^2}{2c}\tfrac{N_+-N_-}{N_++N_-} \mid N_+ + N_- = i\right) \cdot Prob(N_+ + N_- = i) = \\ &= \sum_{i=1}^{N} \left(\tfrac{1+c^2}{2c}\right)^2 \tfrac{1}{i^2}\, Var\left(N_+ - N_- \mid N_+ + N_- = i\right) \cdot Prob(N_+ + N_- = i) \end{aligned} \tag{5.3}$$

where $Var(.)$ denotes the variance of a random variable.

Let be X_j a random variable that takes the value $+1$ if the outcome of the measurement pair is $(+1,+1)$, and -1 when the outcome is $(-1,-1)$. Then $X_j = 1$ with probability $\frac{p_+}{p_++p_-}$, and $X_j = -1$ with probability $\frac{p_-}{p_++p_-}$. These are the conditional properties of being $+1$,$+1$ and -1,-1, if we know that the two outcome is the same. Then

$$Var\left(N_+ - N_- \mid N_+ + N_- = i\right) = Var\left(\sum_{j=1}^{i} X_j\right) = i \cdot Var(X_1).$$

From simple calculation we obtain:

$$Var(X_1) = 1 - \left(\frac{p_+ - p_-}{p_+ + p_-}\right)^2 = 1 - \left(\frac{2cx}{1+c^2}\right)^2$$

Therefore, the variance of the Maximum Likelihood estimator is

$$\mathbf{V}_N(c,x) = \left(\frac{1+c^2}{2c}\right)^2 \left[1 - \left(\frac{2cx}{1+c^2}\right)^2\right] \sum_{i=1}^{N} \frac{1}{i} \cdot Prob(N_+ + N_- = i),$$

$$\text{and} \quad \sum_{i=1}^{N} \frac{1}{i} \cdot Prob(N_+ + N_- = i) = \mathbb{E}\left(\frac{1}{N_+ + N_-}\right) \sim \frac{1}{N(p_+ + p_-)},$$

where $\mathbb{E}$ denotes the mean value, and $\sim$ stands for asymptotic equality. Thus we obtain:

$$\lim_{N\to\infty} N\mathbf{V}_N(c,x) = \frac{(c+1/c)^2 - 4x^2}{2(1+c^2)} = W(c,x)$$

The other important aim would be to minimize the disturbed system instances, i.e. the cases when the outcomes are $(+1,+1)$, or $(-1,-1)$. The probability of having such outcomes is $p(c,x) = \frac{1}{2}(1+c^2)$. Note that $W(c,x)$ can be regarded as the asymptotic variance originating from a qubit, and $p(c,x)$ as the probability that the state of qubit will remain unchanged during the estimation process.

5.2 Optimal measurement strategy

If one wants to have a compromising strategy, then a possible way is to minimize the expression

$$\Psi(c,x) = \min_c \left[\, A \cdot W(c,x) + (1-A) \cdot p(c,x) \,\right],$$

where $A \in \mathbb{R}^+$ is a normalized parameter $(1 \geq A \geq 0)$ which determines our trade-off strategy. If $A \approx 1$, then the aim is accuracy, while in the case of $A \approx 0$ we aim at minimal demolition. Figure 5.1 shows the substantial part of the loss function $\Psi(c, x)$ over the domain $(-1 \leq x \leq 1)$, $(0.2 \leq c \leq 1)$. Note that the function is symmetric to the $c = 0$ line, but it

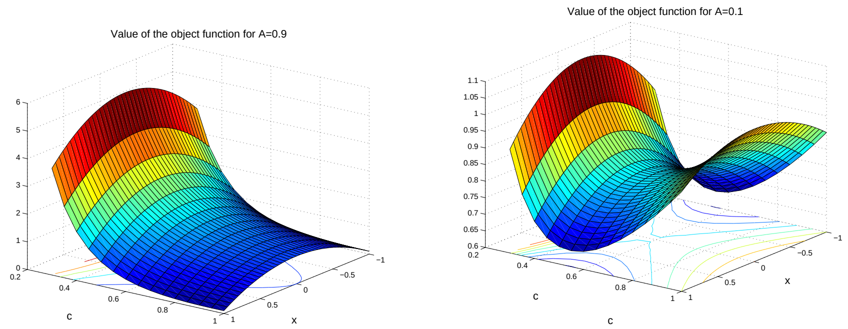

Figure 5.1: The optimal measurement qubit state for different A values: more information ($A = 0.9$, left) versus more non-demolished system ($A = 0.1$, right)

is indefinite at $c = 0$. It is seen that there is a definite optimal value $c \approx 0.6$ for the initial state of the measurement qubit in case $A = 0.1$ that is the same for every x. In the case of $A = 0.9$, however, the minimum is taken at $c = 1$, i.e. at the complete demolition situation.

Chapter 6

Comparison and modifications

In what follows, the indirect estimation scheme defined in Chapter 5 is compared to different other methods. Afterwards, slight modifications are made on it in order to improve its efficiency.

6.1 Comparison with the standard method

It is possible to compare the above estimation scheme with the so-called *standard qubit tomography* [12], which uses information obtained from the von Neumann measurements of the three Pauli matrices.

For the sake of reasonability, it is assumed that the number of measurements m for the standard qubit tomography equals to the expected value of the changed qubits in indirect scheme, i.e. $m = N \cdot p(c, x)$. This way, the expected number of destroyed qubits will be the same in the two compared methods.

The variance of the standard method is $V_m^{stan} = \frac{1}{m} \cdot (1 - x^2)$ if the number of measurements is m. In this case $m = N \cdot \frac{1+c^2}{2}$, so $W^{stan}(c, x) = N \cdot V_m^{stan} = \frac{2(1-x^2)}{1+c^2}$. We can define efficiency η as the quotient of $W(c, x)$ and $W^{stan}(c, x)$, i.e.

$$\eta = \frac{W(c, x)}{W^{stan}(c, x)} = \frac{(c + 1/c)^2 - 4x^2}{2(1 + c^2)} \cdot \frac{1 + c^2}{2(1 - x^2)} = \frac{\frac{1}{4}(c + 1/c)^2 - x^2}{1 - x^2} \geq 1,$$

because $|c + 1/c| \geq 2$, equation holds if $c = \pm 1$.

The above result clearly shows, that the standard method is more accurate than the indirect one. Note, that if $c = \pm 1$, the two methods are the same (see Remark 1), i.e. the standard qubit tomography is the *special case* of the indirect method (see Figure 6.1).

6.2 Modified methods

This section deals with possible modifications of the original method.

6.2.1 Recycling the unchanged qubits

The idea in the following is to use the qubits which remained unchanged during the indirect measurement to obtain more information about the state. So the measurement procedure is

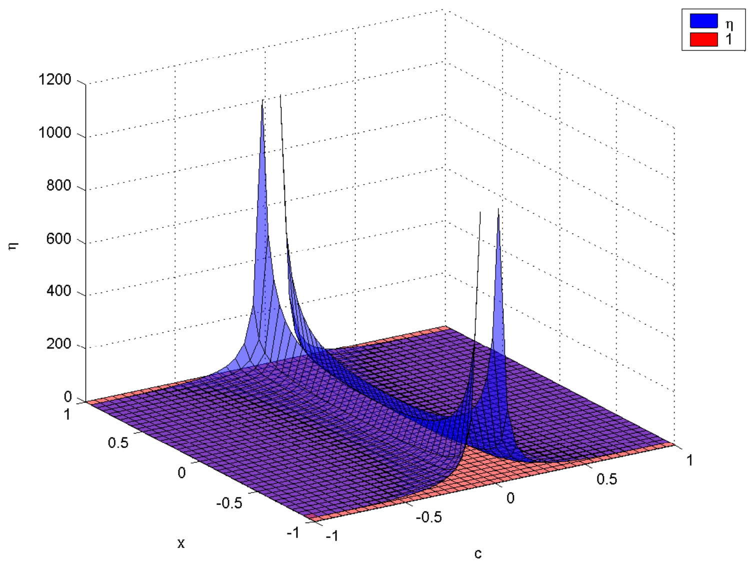

Figure 6.1: The efficiency η (blue) compared to constant one (red).

continued on the unchanged qubits until all of them is changed.

The variance of this modified method comes as the special case of the variance (5.3) of the method defined in section 4.1 with $i = N$. It is easy to see, that

$$\mathbf{V}_N^{recyc}(c,x) = Var\left(\frac{1+c^2}{2c}\frac{N_+ - N_-}{N_+ + N_-} \mid N_+ + N_- = N\right) = \frac{1}{N}\cdot\left[1-\left(\frac{2cx}{1+c^2}\right)^2\right].$$

The variance of standard method for N qubits is $\mathbf{V}_N^{stan}(c,x) = \frac{1}{N}\cdot(1-x^2)$. Computing their quotient, it is

$$\eta^{recyc} = \frac{W^{recyc}(c,x)}{W^{stan}(c,x)} = \frac{\mathbf{V}_N^{recyc}(c,x)}{\mathbf{V}_N^{stan}(c,x)} = \frac{1-\left(\frac{2cx}{1+c^2}\right)^2}{1-x^2} = \frac{1-\left(\frac{x}{\frac{c+1/c}{2}}\right)^2}{1-x^2} \geq 1,$$

since $|c+1/c| \geq 2$. It means, that the indirect method cannot overcome the standard one even in the case when all the available qubits are measured.

However, it is expected that the modified estimation is more effective, than the original one. The fact, that $\eta = \eta^{recyc}\cdot\frac{1}{4}(c+1/c)^2$ supports these expectations, i.e. $\eta \geq \eta^{recyc}$.

6.2.2 Measuring four times

As another way of modification of the method described in Chapter 5, it is possible to change n, i.e. the number of subsequent measurements performed on the coupled qubits. In what follows, n is assumed to be 4. This implies that the number of possible outcomes are $2^4 = 16$, but according to Proposition 1 and Proposition 2 the outcomes like $\{+---\}$, $\{-+--\}$, $\{--+-\}$ and $\{---+\}$ are indistinguishable and correspond to p_{-2} (see Figure 6.2). The

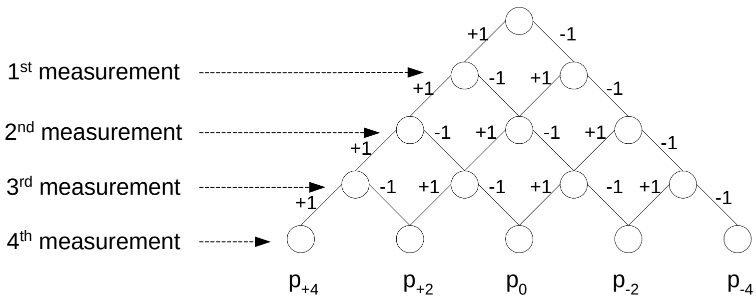

Figure 6.2: The possible outcome-combinations result in five different probabilities $p_{+4}, p_{+2}, p_0, p_{-2}, p_{-4}$.

probabilities of such groups of outcomes can be determined using the result of Corollary 3 multiplied by the appropriate binomial coefficients:

$$\begin{aligned} p_{+4} &= \tfrac{1}{16}\left(1+6c^2+c^4+4c\left(1+c^2\right)x\right) \\ p_{-4} &= \tfrac{1}{16}\left(1+6c^2+c^4-4c\left(1+c^2\right)x\right) \\ p_{+2} &= \tfrac{1}{4}\left(1-c^2\right)\left(1+c^2+2cx\right) \\ p_{-2} &= \tfrac{1}{4}\left(1-c^2\right)\left(1+c^2-2cx\right) \\ p_0 &= \tfrac{3}{8}\left(1-c^2\right)^2 \end{aligned} \tag{6.1}$$

Using the probabilities (6.1) it is easy to construct estimators similar to (5.2):

$$\hat{x}_1(N_2, N_{-2}, c) = \frac{1+c^2}{2c}\frac{N_2-N_{-2}}{N_2+N_{-2}} \tag{6.2}$$

$$\hat{x}_2(N_4, N_{-4}, c) = \frac{1+6c^2+c^4}{4c\left(1+c^2\right)}\frac{N_4-N_{-4}}{N_4+N_{-4}} \tag{6.3}$$

Estimators (6.2) and (6.3) are independent, thus it is possible to use a linear combination of them as an estimation:

$$\hat{x}(N_2, N_{-2}, N_4, N_{-4}, c) = B\cdot\hat{x}_1(N_2, N_{-2}, c) + (1-B)\cdot\hat{x}_2(N_4, N_{-4}, c), \tag{6.4}$$

where $0 \le B \le 1$. In order to have an efficient estimator (6.4), it's variance

$$B^2\cdot Var(\hat{x}_1) + (1-B)^2\cdot Var(\hat{x}_2)$$

should be minimal. The covariance has been omitted because of the independency of estimators (6.2) and (6.3). Similarly to the derivation in subsection 5.1, the variances of the independent estimators are obtained as:

$$W^{\hat{x}_1}(c,x) = \frac{1+c^4+c^2\left(2-4x^2\right)}{2c^2-2c^6}$$

and

$$W^{\hat{x}_2}(c,x) = \frac{1}{2c^2} + \frac{2}{(1+c^2)^2} - \frac{8x^2}{1+6c^2+c^4},$$

using $W = \lim_{N\to\infty} N \cdot Var$. The minimal value of the variance is where its derivative is 0. It is reached with the following value of B:

$$B = \frac{(-1+c^2)\left(-(1+6c^2+c^4)^2 + 16c^2(1+c^2)^2 x^2\right)}{2\left(1+c^2\left(10+24c^2-2c^4-c^6+2(1+c^2)(-5-6c^2+3c^4)x^2\right)\right)}$$

The optimal value of the variance is then

$$W^{four}(c,x) = \frac{(1+c^2-2cx)(1+c^2+2cx)(1+6c^2+c^4-4c(1+c^2)x)(1+6c^2+c^4+4c(1+c^2)x)}{4c^2\left(1+c^2\left(11+34c^2+22c^4-3c^6-c^8+2(1+c^2)^2(-5-6c^2+3c^4)x^2\right)\right)}.$$

Since our other aim is to change as few qubits as possible, it is necessary to write up the ratio of changed qubits. This is simply the complementary probability of being unchanged, i.e.

$$p^{four}(c,x) = 1 - p_0 = 1 - \frac{3}{8}\left(1-c^2\right)^2$$

The efficiency is the following:

$$\eta^{four} = \frac{W^{four}(c,x)\cdot\left(1-\frac{3}{8}(1-c^2)^2\right)}{1-x^2}.$$

Figure 6.3 shows, that the relation $1 \le \eta^{four} \le \eta$ holds, i.e. the modification has slightly improved the performance of the estimator, but it is still worse than the standard one and the estimator formula is much more complicated. It can be seen in the case of both the original and the modified estimator's η, that for a given measurement state with $c \ne \pm 1$, η is minimal, if $x = 0$. This means, that the estimators can achieve smaller variance (better performance) in the case of mixed states.

6.2.3 Undoing the measurement

In this case we use the method described in Section 4.1, perform the estimation, and then we re-measure the changed qubits until they get back to the initial state, i.e. until there is an equal number of measuring +1 and -1 outcomes.

Each path that returns in $2k$ steps to the origin has the probability $\left(\frac{1-c^2}{4}\right)^k$ (the number of + and - outcomes is both k). At the same time the number of pathes that return to the initial state first in the $2k$-th step is exactly $2\cdot C_{k-1}$, where C_k is the k-th Catalan number. Let us use the notation $r = \frac{1-c^2}{4}$, so the probability that the state will ever return to the initial state, i.e we can undo the measurement, is

$$1 - p^{undo}(c,x) = \sum_{k=1}^{\infty} 2\cdot C_{k-1}\, r^k = 2\sum_{k=0}^{\infty} C_k\, r^{k+1} = 2r\sum_{k=0}^{\infty} C_k\, r^k.$$

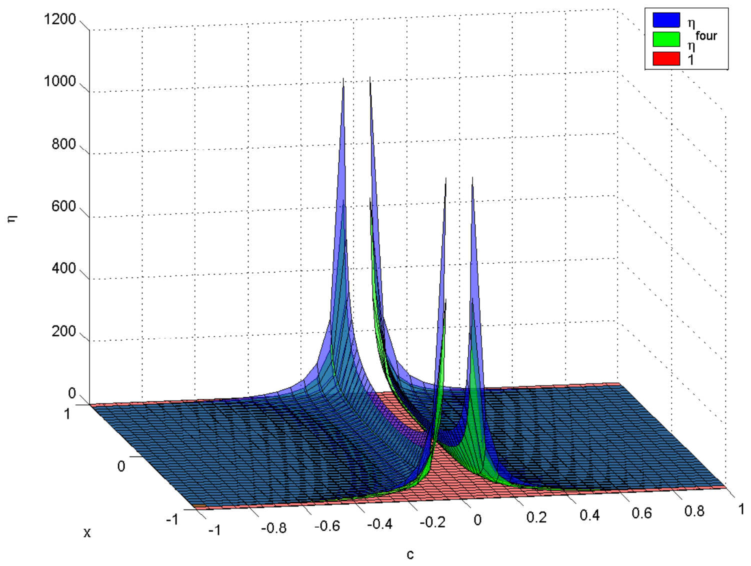

Figure 6.3: The quotients η (blue) and η^{four} (green) compared to constant one (red).

The finally received sum is the generating function of Catalan numbers, and it has a well known closed formula:

$$\sum_{k=0}^{\infty} C_k \ r^k = \frac{1-\sqrt{1-4r}}{2r}$$

So

$$1 - p^{undo}(c,x) = 2r \cdot \frac{1-\sqrt{1-4r}}{2r} = 1 - \sqrt{1-4r} = 1 - \sqrt{1-4\frac{1-c^2}{4}} = 1 - |c|$$

and we get that the probability that the qubit never returns $p^{undo}(c,x) = |c|$. So according the previous section ($W^{undo}(c,x) = W(c,x)$) the efficiency is the following (see also Figure 6.4):

$$\eta^{undo} = \frac{(c+1/c)^2 - 4x^2}{2(1+c^2)} \cdot \frac{|c|}{1-x^2}$$

The greatest improvement is at small c values, but the original efficiency (η)is the worst exactly here. Therefore, the situation is like in previous cases: we have achieved an improvement in the efficiency but it is still under the efficiency of the standard method, i.e. $1 < \eta^{undo} < \eta$ (see Figure 6.4). Note that this calculation gives back the trivial fact, that in the standard case ($c = 1$) we never return to the original state: $p^{undo} = 1$.

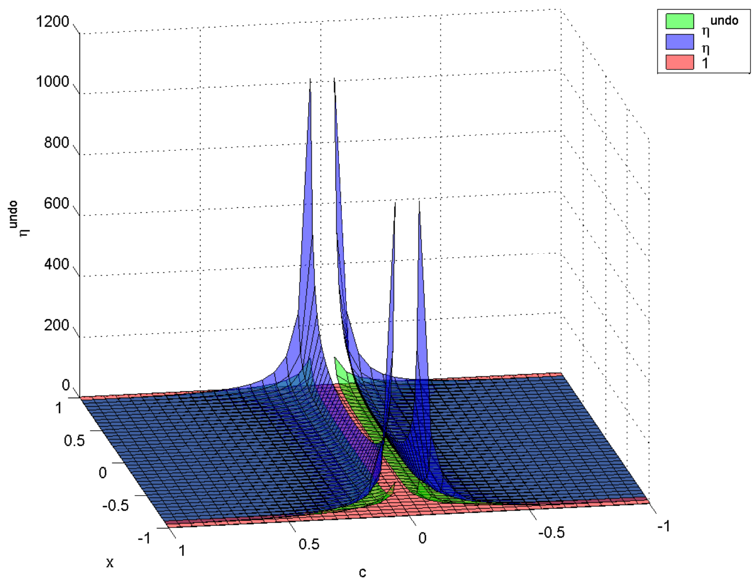

Figure 6.4: The efficiency η^{undo}

Chapter 7

Conclusions

Indirect qubit tomography methods that allows us to find a trade-off between the effectiveness of the estimator and the number of destroyed state copies are investigated in this work. As opposed to the majority of the existing results in this field, a discrete-time approach is applied to the indirect measurement.

The simplest possible case has been considered, where both the unknown and the measurement quantum systems are quantum bits. The measurements applied on the measurement qubit have been the classical von Neumann measurements using the Pauli matrices as observables.

The statistical properties of the estimate in terms of the variance of the ML estimator and the non-demolition probability have been analytically calculated in the case of subsequent measurements applied in the x direction while the qubits interact in the y direction and the initial Bloch-vector of the measurement qubit is $[0, 0, c]$. A way of finding an optimal compromising measurement strategy between the asymptotic variance and the non-demolition probability has been proposed. The efficiency of the results have been compared with a classical 'standard' state estimation procedure available in the literature. Although the classical one performs better by means of the variance, the indirect one gives a degree of freedom in the above mentioned trade-off problem. The estimation method has also been modified in a few ways to improve its precision. It has been shown that the modified measurement strategies may reach the efficiency of the standard method in the limit when the complete demolition situation is achieved. This is in good agreement with the results of D'Ariano and Yuen [2].

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