

Improved Estimation Method of Region of Stability for Nonlinear Autonomous Systems

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Abstract

The stability and the stability region of a (controlled) industrial system is an important property to be determined. It is highly desirable to have a system that is not able to become unstable that is why the stability theory is being continuously in both the theoretical researchers's and industrial partitioner's focus. The aim of this paper is to examine the stability region of an existing industrial plant of great importance using stability region estimation methods can be found in literature (see [1],[2],[3],[4]).

Contents

1	Introduction	2
2	Basic notions	2
3	Maximal Lyapunov functions	2
4	Computation of the Domain of Attraction (DOA)	4
5	Properties of the recursive algorithm	6
6	Case studies	7
6.1	Van der Pole-equation	7
6.2	Lotka-Volterra	8
6.3	Fermentation system	8
7	Conclusion and Future work	8

List of Figures

1	Pairs of Q_i and R_i which should be multiplied in Equation 5.8	10
2	Domain of attraction of the Van der Pole system	11
3	Direct check of stability of the Van der Pole-equation	11
4	Domain of attraction of the Lotka-Volterra system	12
5	Direct check of stability of the Lotka-Volterra system	12
6	Stability region of the fermentation system	13

1 Introduction

In this article an improved algorithm will be shown to estimate the region of attraction of nonlinear autonomous systems. The concept is based on [3]. The algorithm will also be analyzed and a number of examples will be shown.

2 Basic notions

In the following we will consider the nonlinear autonomous system supposing that the origin is its asymptotically stable equilibrium point:

$$\dot{x}(t) = f(x(t)) \quad (2.1)$$

By the region or domain of attraction $S(M)$ of the set M (which need not to be an attractor) we mean the union of all trajectories with the property that their limit sets are non-empty and contained by M itself. Based on this composition we can define the domain of attraction of the origin as a set having only one element.

Definition 2.1. *The domain of attraction of the origin is the set*

$$S(0) = \{x_0 : x(t, x_0) \rightarrow 0 \text{ as } t \rightarrow \infty\} \quad (2.2)$$

where $x(t, x_0)$ denotes the solution of the system in Equation 2.1 corresponding to the initial condition $x(0) = x_0$.

In this paper it is shown by following the ideas of [3] that there exists a sequence of special kind of Lyapunov functions V_m that can be used to estimate the set $S(0)$ through estimating a Lyapunov function of special kind. An iterative method will be given to find these appropriate V_m s. The given algorithm is able to find unbounded domains of attraction, too. Usually, the first few number of iterations can show if the domain is bounded or not.

Throughout the paper it is assumed that the function f is smooth enough that Equation 2.1 has unique solution corresponding to each initial condition $x(0) = x_0$. If the solution is Lyapunov-stable then it depends continuously on x_0 . Being stable at t_0 concludes stability at any other t_0' , too. According to a known theorem in [4] if the function f does not depend on t or periodic in t then the stability of the origin implies uniform stability and asymptotical stability implies uniform asymptotical stability. For the definitions of different types of attraction and stability see for example [5].

3 Maximal Lyapunov functions

The proofs of the statements in this section can be found in [3].

Theorem 3.1. *Suppose we can find a set $A \subseteq \mathbb{R}^n$ containing the origin in its interior, a continuous function $V : A \rightarrow \mathbb{R}_+$ and a positive definite function ϕ such that*

1. $V(0) = 0, V(x) > 0 \forall x \in A \setminus \{0\}$

2. *The function*

$$\dot{V}(x_0) = \lim_{t \rightarrow 0^+} \frac{V(x(t, x_0)) - V(x_0)}{t} \quad (3.1)$$

is well defined at all $x \in A$ and satisfies

$$\dot{V}(x) = -\phi(x), \forall x \in A. \quad (3.2)$$

3. $V(x) \rightarrow \infty$ as $x \rightarrow \partial A$ and/or $\|x\| \rightarrow \infty$.

Then $A = S$.

Corollary 3.2. *Suppose we can find a set $A \subseteq \mathbb{R}^n$ containing the origin in its interior, a continuously differentiable function $V : A \rightarrow \mathbb{R}_+$ and a positive definite function ϕ such that*

1. $V(0) = 0, V(x) > 0 \forall x \in A \setminus \{0\}$
2. $\nabla V(x)' f(x) = -\phi(x) \forall x \in A$
3. $V(x) \rightarrow \infty$ as $x \rightarrow \partial A$ and/or $\|x\| \rightarrow \infty$.

Then $A = S$.

Definition 3.3. *We say that a function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class K if ϕ is continuous, $\phi(0) = 0$ and ϕ is monotonically increasing.*

The next results show that the conditions on V imposed in Theorem 3.1 are reasonable.

Theorem 3.4. *Suppose f is continuously differentiable in some neighborhood of 0. Then there exists a continuous function $V_m : S \rightarrow \mathbb{R}_+$ and $\gamma \in K$ such that*

1. $V_m(0) = 0, V_m(x) > 0 \forall x \in S \setminus \{0\}$
2. $\dot{V}_m(x) = -\gamma(\|x\|) \forall x \in S$
3. $V_m(x) \rightarrow \infty$ as $x \rightarrow \partial S$. Moreover, if f is Lipschitz continuous on S then V_m can be selected to be continuously differentiable on S and
4. $V_m(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

Theorem 3.5. *Suppose f is Lipschitz-continuous on S . Then in order for an open set A containing the origin to be the domain of attraction for Equation 2.1, it is necessary and sufficient that there exist a continuous function $V : A \rightarrow \mathbb{R}_+$ and a positive definite function ϕ such that conditions of Theorem 3.1 hold true.*

Suppose V is a continuous function on some ball B_δ such that $V(0) = 0$ and \dot{V} is negative definite. Then one could prove that V is positive definite. This fact shows that if we can find a function V and a positive definite function ϕ such that $V(0) = 0$ and

$$\nabla V(x)' f(x) = -\phi(x) \tag{3.3}$$

then V is guaranteed to be positive definite.

Definition 3.6. *A function $V_m : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is called maximal Lyapunov function for the system described in Equation 2.1 if*

1. $V_m(0) = 0, V_m(x) > 0, x \in S \setminus \{0\}$
2. $V_m(x) < \infty \Leftrightarrow x \in S$
3. $V_m(x) \rightarrow \infty$ as $x \rightarrow \partial S$ and/or $\|x\| \rightarrow \infty$
4. \dot{V}_m is well-defined and negative definite over S .

Remark 3.7. *Theorem 3.4 shows that a maximal Lyapunov function exist under not too strict conditions on f .*

4 Computation of the Domain of Attraction (DOA)

We look for a function V and a positive definite function ϕ satisfying $V(0) = 0$ and

$$\dot{V}(x) = -\phi(x) \quad (4.1)$$

over some neighborhood of 0. Then the boundary of the domain of attraction is defined by the limit $V(x) \rightarrow \infty$.

A systematic procedure will be discussed here to solve Equation 4.1 supposing that the Taylor series expansion exists for f around the origin. Express f as

$$f(x) = \sum_{i=1}^{\infty} F_i(x) \quad (4.2)$$

where the functions $F_i, i \geq 1$ are homogeneous functions of degree i . For $i = 1$ we have

$$F_1(x) = Ax, A \in \mathbb{R}^{n \times n}. \quad (4.3)$$

For the sake of brevity let $F_i(x) = 0, i \leq 0$.

Our candidate Lyapunov function should excess any limit as x gets closer to the boundary of set S or as $\|x\| \rightarrow \infty$. For this reason we put $D(x)$ to the denominator, i.e.

$$V(x) = \frac{N(x)}{D(x)} \quad (4.4)$$

where $N(x)$ and $D(x)$ are polynomials in x . Thus, $V(x) \rightarrow \infty$ as $x \rightarrow \partial S$ and this suggests that $x \in \partial S$ when $D(x) = 0$. According to the results of Section 3 the boundary of S is defined by solving $D(x) = 0$ for x . We obtain a recursive technique to find this boundary by defining

$$V(x) = \frac{\sum_{i=2}^{\infty} R_i(x)}{1 + \sum_{i=1}^{\infty} Q_i(x)} \quad (4.5)$$

where R_i and Q_i are homogeneous functions of degree i . The most straightforward idea for ϕ is $x' \Omega x$ where $\Omega > 0$. Substituting this expression into Equation 4.1 we obtain

$$\dot{V}(x) = \nabla V(x)' f(x) = -\phi(x) = -x' \Omega x \quad (4.6)$$

Based on Equation 4.5 and Equation 4.6 we get

$$\left(\left(1 + \sum_{i=1}^{\infty} Q_i \right) \sum_{i=2}^{\infty} \nabla R_i' - \left(\sum_{i=1}^{\infty} \nabla Q_i' \right) \sum_{i=2}^{\infty} R_i \right) \sum_{i=1}^{\infty} F_i = -x' \Omega x \left(1 + \sum_{i=1}^{\infty} Q_i \right)^2. \quad (4.7)$$

From this finally we obtain

$$\sum_{i=2}^{\infty} \sum_{k=1}^{\infty} \nabla R_i' F_k + \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} Q_i \nabla R_j' F_k - \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} Q_i' R_j F_k = -x' \Omega x \left(1 + 2 \sum_{i=1}^{\infty} Q_i + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} Q_i Q_j \right). \quad (4.8)$$

Equating the coefficients of the same degrees of the two sides of Equation 4.8 we get for degree 2 that

$$\nabla R_2' F_1 = -x' \Omega x \quad (4.9)$$

and the general solution when degree k is greater than or equal to 3 is

$$\sum_{i=2}^k \nabla R_i' F_{k+1-i} + \sum_{i=1}^{k-2} \sum_{j=2}^{k-1} (Q_i \nabla R_j' - \nabla Q_i' R_j) F_{k+1-i-j} = -x' Q x \left(2Q_{k-2} + \sum_{i=1}^{k-3} Q_i Q_{k-2-i} \right). \quad (4.10)$$

Thus in each step of the algorithm we get the following linear under determined set of equations as the equivalent form of the previous two equations:

$$A_n y = b_n \quad (4.11)$$

where A_n are matrices of appropriate dimension. Consider the nonlinear system of equations

$$\dot{x} = f(x) = \sum_{i=1}^{\infty} F_i(x). \quad (4.12)$$

First, select homogeneous functions R_n and $Q_{n-2}, n \geq 3$, such that the coefficients of R_n and Q_n solve the constrained minimization problem yielded by Equation 4.11

$$\begin{aligned} \min e_n(y) \\ \text{s.t. } A_n(y) = b_n \end{aligned} \quad (4.13)$$

where $e_n(y)$ is the square of 2-norm of the coefficients of degree greater than or equal to $n + 1$ in the expression of \dot{V}_n . Furthermore, according to the theorem of La-Salle about invariant sets one can choose the largest positive value C^* such that the level set

$$V_n = \frac{\sum_{i=2}^n R_i}{1 + \sum_{i=1}^{n-2} Q_i} = C^* \quad (4.14)$$

is contained in the region given by

$$\Omega = \left\{ x : \dot{V}_n(x) \leq 0 \right\}. \quad (4.15)$$

Then the set

$$S_A = \{x : V_n(x) < C^*\} \quad (4.16)$$

is contained in the region of attraction S . If this is the case then the iteration should be stopped as soon as the desired accuracy has been reached.

If $e_n(y^*) = 0$ for some y^* then the iteration can be stopped and

$$\dot{V}_n = -x' \Omega x \quad (4.17)$$

where $\Omega > 0$. In this case the domain of attraction is defined by the equation

$$D(x) = 0. \quad (4.18)$$

This means that the domain of attraction S is given by the formula

$$S = \left\{ x : \sum_{i=1}^{n-2} Q_i > -1 \right\}. \quad (4.19)$$

Would any of the afore-mentioned cases happen for the major part of the systems the iteration can be stopped less then 10 cycles.

5 Properties of the recursive algorithm

Let us show that the sequence of functions V_n are Lyapunov-functions. The result in this section is the corrected form of the one found in the original publication [3].

Theorem 5.1. *Consider the system of nonlinear equations in Equation 4.12. Assume that the linearized system*

$$\dot{x} = F_1(x) = Ax \quad (5.1)$$

is asymptotically stable. Let the homogeneous functions R_i and Q_i satisfy the following recursive equations for second order terms

$$\nabla R_2' F_1 = -x' \Omega x \quad (5.2)$$

and for terms of degree greater than 2

$$\begin{aligned} \nabla R_2' F_{k-1} + \sum_{j=3}^k \left(\nabla R_j' + \sum_{i=1}^{j-2} (Q_i \nabla R_{j-i}' - \nabla Q_i' R_{j-i}) \right) F_{k-j+1} = \\ - x' \Omega x \left(2Q_{k-2} + \sum_{i=1}^{k-3} Q_i Q_{k-2-i} \right), k \geq 3 \end{aligned} \quad (5.3)$$

where Q is a fixed positive definite matrix. Then

$$V_n = \frac{R_2(x) + R_3(x) + \dots + R_n(x)}{1 + Q_1(x) + \dots + Q_{n-2}(x)} \quad (5.4)$$

is a Lyapunov-function for all $n \geq 2$.

Proof. The proof is slightly different from the one found in [3]. Let

$$U = \left(1 + \sum_{i=1}^{n-2} Q_i(x) \right)^2 \quad (5.5)$$

Then

$$\begin{aligned} \dot{V}_n = \nabla V_n' f &= \frac{1}{U} \left(\left(1 + \sum_{i=1}^{n-2} Q_i \right) \sum_{j=2}^n \nabla R_j' - \left(\sum_{i=1}^{n-2} \nabla Q_i' \right) \left(\sum_{j=2}^n R_j \right) \right) f \\ &= \frac{1}{U} \left(\sum_{i=2}^n \nabla R_i + \sum_{i=1}^{n-2} \sum_{j=2}^n (Q_i \nabla R_j' - \nabla Q_i' R_j) \right) \sum_{k=1}^{\infty} F_k = \frac{1}{U} (\dot{V}_{n\text{main}} + \dot{V}_{n\text{resid}}) \end{aligned} \quad (5.6)$$

where

$$\dot{V}_{n\text{main}} = \nabla R_2' F_1 + \sum_{k=3}^n \left(\nabla R_2' F_{k-1} + \sum_{j=3}^k \left(\nabla R_j' + \sum_{i=1}^{j-2} (Q_i \nabla R_{j-i}' - \nabla Q_i' R_{j-i}) \right) F_{k-j+1} \right) \quad (5.7)$$

and

$$\dot{V}_{n\text{resid}} = \sum_{i=1}^{\infty} \left(\sum_{j=n-i+1}^{n-1} \nabla R_{j+1} + \sum_{r=2}^n \sum_{q=(n+1)-(i-1)-r}^{n-2} (Q_q \nabla R_r' - \nabla Q_q' R_r) \right) F_i - \nabla R_1' \sum_{i=n+1}^{\infty} F_i. \quad (5.8)$$

For simpler notation let Q_i, R_i and F_i be zeros if $i \leq 0$. To understand the structure of $\dot{V}_{n\text{resid}}$ you can consider that the index combinations in the double-sum have characteristic structure as you can see it on Figure 1 where $n = 5$. Note that pairs with index $i > n$ should not be taken as summands. Moreover an important property of $\dot{V}_{n\text{resid}}$ is that it contains terms only of degree greater than or equal to $n + 1$. Substituting the left hand side of Equation 5.2 and Equation 5.3 into Equation 5.7 we obtain

$$\begin{aligned} \dot{V}_n &= \frac{1}{U} \left(-x' \Omega x - x' \Omega x \left(2 \sum_{k=3}^n Q_{k-2} + \sum_{k=3}^n \sum_{i=1}^{k-3} Q_i Q_{k-2-i} \right) \right) + \frac{1}{U} \dot{V}_{n\text{resid}} \\ &= \frac{1}{U} \left(-x' \Omega x \left(1 + 2 \sum_{k=3}^n Q_{k-2} + \sum_{k=3}^n \sum_{i=1}^{k-3} Q_i Q_{k-2-i} \right) \right) + \frac{1}{U} \dot{V}_{n\text{resid}} \quad (5.9) \end{aligned}$$

Because the numerator of the first term is subexpression of U with degree up to and including $n - 2$ Equation 5.9 can be written in the form of

$$\dot{V}_n = \frac{1}{U} \left(-x' \Omega x (U - U_g) + \dot{V}_{n\text{resid}} \right) = -x' \Omega x + \frac{x' \Omega x U_g + \dot{V}_{n\text{resid}}}{U} \quad (5.10)$$

where U_g is the sum of terms in U of degree greater than or equal to $n - 1$. This shows that \dot{V}_n is negative definite around the origin. \square

6 Case studies

In this section three examples will be shown. In the first and second examples the minimization problem in Equation 4.13 cannot be solved such that $e(n)$ becomes small enough that we could apply Equation 4.19. Instead, Equation 4.16 is used for the estimation for the domain of attraction. In the last example the minimization problem can be solved and one can use Equation 4.19 to get a proper region.

In the pictures showing the level sets (Figure 2 and Figure 4) the red colour curve shows $V = C^*$ while the blue one shows those points where $\dot{V} = 0$.

6.1 Van der Pole-equation

The Van der Pole system we took as example is described with the equation-system

$$\begin{aligned} \dot{z}_1 &= -z_2 \\ \dot{z}_2 &= z_1 - z_2 + z_1^2 z_2 \end{aligned}$$

The origin is an asymptotically stable equilibrium point of the system thus the method based on maximal Lyapunov functions can be used in this case. Applying the iteration steps 9 times we get $C^* = 6.6$ and the the minimum value of $e(n)$ was found to be 0.0168945. The stability region is the innermost area bounded by the red colored curve in Figure 2. We stopped the iteration at step 9 because further steps could not increase the minimum significantly.

To verify the given region we used a direct method by scanning the points over the region $[-3, -3] \times [-3, -3]$ and examining if the system remains stable or not. The resulted set can be seen in Figure 3. We can ascertain that the regions found by the two different methods are pretty the same.

6.2 Lotka-Volterra

For this example we took a Lotka-Volterra system described by the equations

$$\begin{aligned} \dot{z}_1 &= -z_1((z_1 - 1)(z_1 - 3) + \frac{1}{2}z_2) \\ \dot{z}_2 &= z_2(-2.1 + z_1) \end{aligned}$$

Both the origin and point (2.1, 1.98) both are equilibria of the system. By shifting the second equilibrium to the origin we get the following centralized system

$$\begin{aligned} \dot{x}_1 &= -0.42x_1 - 2.3x_1^2 - x_1^3 - 1.05x_2 - \frac{1}{2}x_1x_2 \\ \dot{x}_2 &= 1.98x_1 + x_1x_2 \end{aligned}$$

Just like the previous case after 6 steps of the algorithm we get $C^* = 1.65$ and the minimum of $e(n)$ was 2915.040375. See the region in Figure 4 bounded by the inner red curve.

By scanning the points over the region $[-1, 1.5] \times [-1.5, 1.5]$ we find that the region estimated by the Vanelli-method is a subset of the one we found by direct search, see Figure 5.

6.3 Fermentation system

In this example we show a simple model of a fermentation system [6] described by the equations

$$\begin{aligned} \dot{z}_1 &= -0.802228z_1 + \frac{z_1z_2}{0.03 + z_2 + 0.5z_2^2} \\ \dot{z}_2 &= 0.802228(10 - z_2) - \frac{z_1z_2}{0.03 + z_2 + 0.5z_2^2} \end{aligned}$$

which has one asymptotically stable equilibrium point: (4.89067, 0.218662). Shifting the system by this point we get the transformed equation system

$$\begin{aligned} \dot{x}_1 &= -0.802228(4.49067 + x_1) + \frac{2.13881 + 0.437324x_1 + 9.79134x_2 + 2x_1x_2}{0.545137 + 2.43732x_2 + x_2^2} \\ \dot{x}_2 &= 7.84686 - 0.802228x_2 + \frac{-4.27762 - 0.874648x_1 - 19.5627x_2 - 4x_1x_2}{0.545137 + 2.43732x_2 + x_2^2} \end{aligned}$$

After 3 steps of the algorithm the minimalization problem can be solved that $e(n)$ becomes zero that is why we can apply Equation 4.19 and we get the region of stability as it is seen in Figure 6. We can see that the given region is very small and thus meets our expectations as it is described in [6].

7 Conclusion and Future work

In this report an improved algorithm based on constructing maximal Lyapunov functions [3] is proposed to estimate the region of attraction of nonlinear autonomous systems. The advantage of this algorithm is that one does not have to know the solution of the system starting from different initial values; only a minimization problem (a linear programming problem) needs to be solved in each step of the recursive approximation procedure. Moreover, the applicable system class is wider than that of the majority of available algorithms can handle (they are mainly restricted to polynomial systems). And lastly, this algorithm is faster than the well-known Zubov's method as only a few steps are needed instead of a few times ten. In addition, it is accurate enough that one could use it instead of the cumbersome (however more exact) scanning approach.

A few problems arose during the analysis and implementation of the described algorithm, they can be divided into two main area. One of them is the deeper investigation of the method to find broader class of systems which can be analyzed, for example, the class of periodic non-autonomous systems or the so-called partial stable systems (for definitions see [5]) are promising. An other important problem is to seek for restrictive conditions under which the number of iterations could be decreased or at least estimated.

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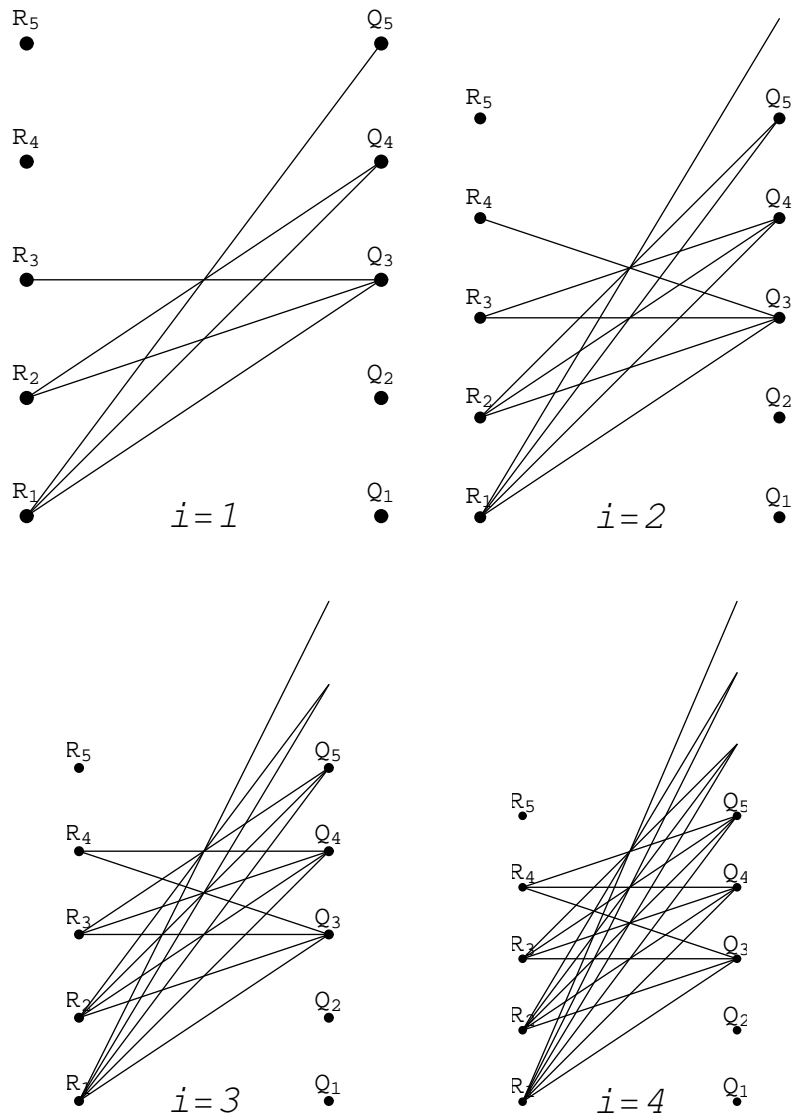


Figure 1: Pairs of Q_i and R_i which should be multiplied in Equation 5.8

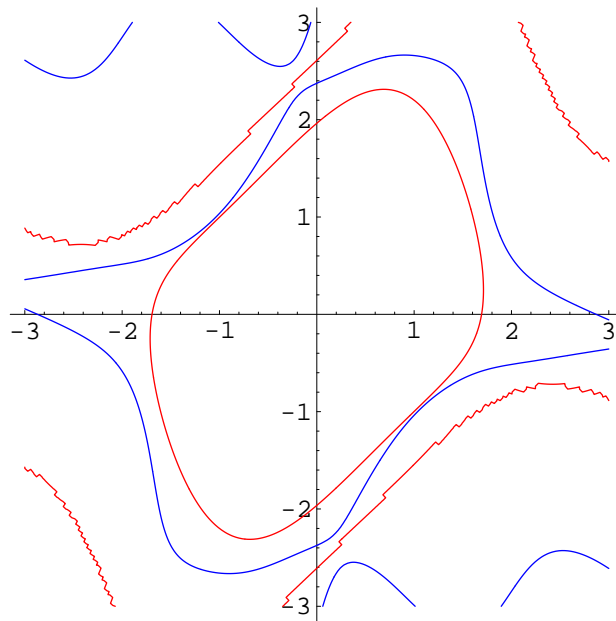


Figure 2: Domain of attraction of the Van der Pole system

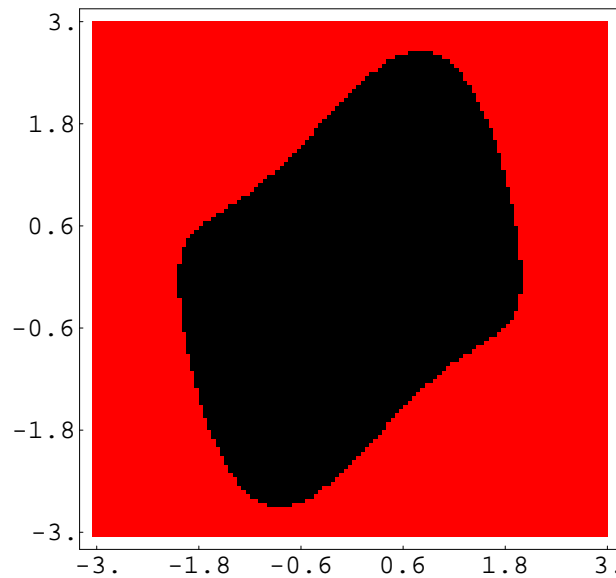


Figure 3: Direct check of stability of the Van der Pole-equation

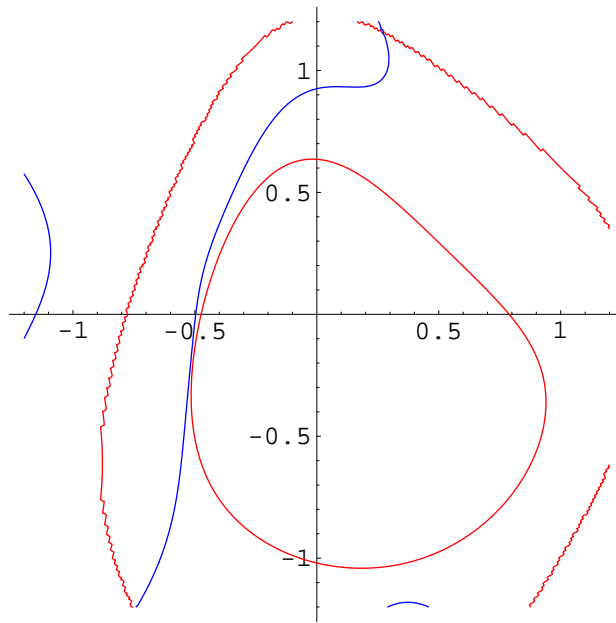


Figure 4: Domain of attraction of the Lotka-Volterra system

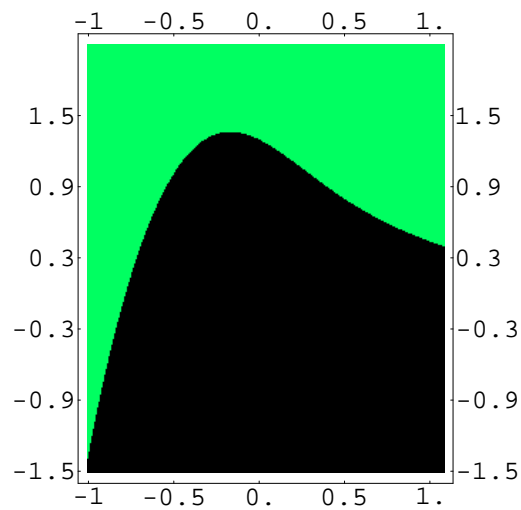


Figure 5: Direct check of stability of the Lotka-Volterra system

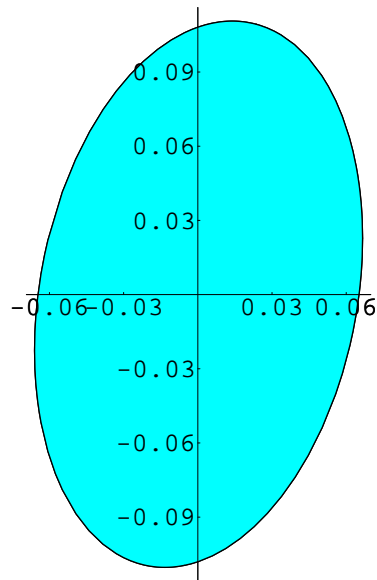


Figure 6: Stability region of the fermentation system