State estimation methods using indirect measurements

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Abstract

The report presents mathematically well grounded statistical methods for state estimation in the indirect measurement setting, when the measurement is performed on an ancilla system that is put into interaction with the unknown one.
The considered measurement scheme is the simplest possible discrete time case, where both the unknown and the ancilla quantum systems are quantum bits. The measurements applied on the ancilla qubit are the classical von Neumann measurements using the Pauli matrices as observables. The repeated measurements performed on the ancilla enables us to construct estimators of the initial state of the unknown system. Based on the statistical properties of the considered indirect measurement scheme [15], three related but different approaches are proposed and investigated: (i) a direct estimation procedure that is based on the estimated relative frequencies of the characterizing conditional probability densities, (ii) Bayesian recursive approach for state estimation, and (iii) a martingale approach that bases the estimator on the stopping times of the state evolution as a martingale driven by the repeated measurements.
The statistical properties, i.e. the unbiasedness and the efficiency of the proposed procedures are investigated both analytically and experimentally using simulation.

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Chapter 1

Introduction

It is well known that a projective measurement applied to a quantum system will change the state of the measured system in an irreversible way depending on the measurement outcome [11]. In addition, the probabilistic nature of the measurement result calls for applying a state estimation (or state tomography) approach if one wants to have information about the state of the quantum system.

Similarly to any realistic physical measurement, a quantum measurement is almost always realized by taking a measurement device that is put in interaction with the system to be measured, and then to "read" the meter on the measurement device. In the macroscopic measurement situation the measurement device is "small" compared to the system to be measured, thus the measurement back-action, i.e. the disturbance caused by the measurement is negligible, but that is not the case in the quantum setting. In quantum state estimation the above measurement configuration, when the 'unknown' quantum system is coupled with a 'measurement' (also called *'ancilla'*) system, and the measurements are only applied on the ancilla system [6] is termed an *indirect measurement scheme*. In the field of solid-state quantum bits a not fully realistic but conceptually simple model of indirect projective measurement is when the measured qubit interacts with another (ancillary) qubit, which is later measured in the "orthodox" projective way [10].

The notion of *weak measurement*s is related to the notion of indirect measurements, but approaches the measurement back-action problem in a different way. A weak measurement [16, 4] is designed not to demolish the system state completely, i.e. the post-measurement state still contains information about the original one, but on the price of a decreased information gained from the measurement. In the weak measurement setting the measured variable has an effective interaction with the unknown one "in the limit of weak coupling" thus minimizing the disturbance caused (and the information gained) by the measurement.

Besides of state estimation, weak measurements are used for other related tasks, such as state purification or noise reduction combined with suitable feedback, see e.g. [1], [5]. In a particularly interesting paper Korotkov and Jordan [9] have shown that "it is possible to fully restore any unknown, pre-measured state, though with probability less than unity" for solid-state qubits and continuous time measurements. Recently, a similar approach for reversing the weak quantum measurement for a photonic qubit has also been reported [8].

However, it is intuitively clear, that one must make a compromise between the information gained in a measurement and the disturbance or demolition caused by it. The general impossibility of determining the state of a single quantum system is proved in [2] whatever

measurement scheme is used. This indicates that the efficiency or precision provided by an indirect measurement scheme is necessarily smaller than that of a scheme that uses von Neumann measurements.

The aim of this work is to propose mathematically well grounded statistical methods for for state estimation for the indirect measurement setting and compare their efficiency to the usual direct approaches.

Chapter 2

The investigated indirect measurement scheme

The ingredients of the investigated indirect measurement scheme, the quantum system, the observables, and the measurement strategy are described following the work [15].

2.1 State representation, interaction, time evolution and observables

The investigated composite system consists of two qubits that are put into interaction, and then a measurement is performed on one of them, termed the ancilla qubit.

State representation Throughout the paper the *Bloch-vector* representation of the states of quantum bits is used. The system to be investigated consists of two qubits in interaction: the unknown system (subscript $_S$) and the ancilla (subscript $_M$) qubits, their Bloch representation is in the form

$$\rho_S(k) = \frac{1}{2}(I + \theta_S(k)\sigma^S) \quad , \quad \rho_M(k) = \frac{1}{2}(I + \theta_M(k)\sigma^M), \tag{2.1}$$

where θ_S and θ_M are 3 dimensional real vectors, σ^S and σ^M are symbolic vectors constructed from the Pauli operators acting on the Hilbert spaces $\mathcal{H}^S$ and $\mathcal{H}^M$.

The state of the composite system is represented as a 4×4 density matrix $\rho_{S+M}(k)$. The state of the composite system after the interaction is given by

$$\rho_{S+M}(k+1) = U_{S+M}\rho_{S+M}(k)U^*_{S+M} \tag{2.2}$$

where U_{S+M} is the overall system evolution unitary. Since we are interested in the dynamical change of the system S, the first reduced density matrix should only be considered:

$$\rho_S(k+1) = \mathrm{Tr}_1\rho_{S+M}(k+1) = \mathrm{Tr}_M\rho_{S+M}(k+1). \tag{2.3}$$

Interaction In order to have a simple parametrization of the interaction (coupling) between the unknown and ancilla qubits, the Cartan decomposition [7, 13] of the discrete time evolution unitary U_{S+M} is used in the form

$$U_{S+M} = L_1 e^{ah} L_2 \tag{2.4}$$

where L_1 and L_2 are in $SU(2) \otimes SU(2)$ and $a \in \mathbf{a}$ with

$$\mathbf{a} = \mathrm{i}\ span\{\sigma_1^S \otimes \sigma_1^M, \sigma_2^S \otimes \sigma_2^M, \sigma_3^S \otimes \sigma_3^M\} \tag{2.5}$$

Because both L_1 and L_2 are in a product form, they describe the product of the local dynamical effects L_i^S and L_i^M $(i = 1, 2)$, and the interaction is parameterized by three real parameters a_1, a_2 and a_3.

Therefore, the dynamical equation of qubit S in (2.3) becomes

$$\rho_S(k+1) = L_1^S \mathrm{Tr}_M \left(e^{ah} \tilde{\rho}_S(k) \otimes \tilde{\rho}_M(k) e^{a^* h} \right) L_1^{S*} \tag{2.6}$$

where $L_1 = L_1^S \otimes L_1^M$, $L_2 = L_2^S \otimes L_2^M$ both time dependent, and $\tilde{\rho}_S = L_2^S \rho_S L_2^{S*}$, $\tilde{\rho}_M = L_2^M \rho_M L_2^{M*}$. In order to simplify the forthcoming computations, we consider the case *with no local dynamics*, when $L_i^S = L_i^M = I$ $(i = 1, 2)$.

Observables The most generally used von Neumann measurement is the measurement of the Pauli operators σ_1, σ_2, or σ_3. If one considers the measurement of the observable σ_1, then the possible outcomes are the different eigenvalues of the observable, i.e. ± 1. The probabilities of the different outcomes are

$$\begin{aligned} \mathrm{Prob}(+1) &= \mathrm{Tr}\rho E_{+1} = \tfrac{1}{2}(1 + \theta_1) \\ \mathrm{Prob}(-1) &= \mathrm{Tr}\rho E_{-1} = \tfrac{1}{2}(1 - \theta_1) \end{aligned}$$

respectively, where the spectral decomposition of σ_1 is $\sigma_1 = E_{+1} - E_{-1}$ with

$$E_{\pm 1} = (I \pm \sigma_1)$$

2.2 Direct state estimation of qubits

The performance of the proposed methods in the indirect setting will be compared to that of the direct state estimation methods using the same observables and projective measurements.

The most widespread and convenient method for estimating the state of a qubit is to use direct measurements on multiple copies of identically prepared qubits. In our case this corresponds to the situation when the measurements are performed on the unknown system. The so called standard scheme applies the Pauli matrices $\sigma_i,\ i = 1, 2, 3$ as observables. Then the estimator for the Bloch vector θ_S of the qubit is

$$\Phi_{\mathbf{n}} = \begin{bmatrix} 2\nu(n_1, \sigma_1, +1) - 1 \\ 2\nu(n_2, \sigma_2, +1) - 1 \\ 2\nu(n_3, \sigma_3, +1) - 1 \end{bmatrix} \tag{2.7}$$

when σ_i is measured n_i times with the relative frequency of its $+1$ outcome $\nu(n_i, \sigma_i, +1)$, $1 \leq i \leq 3$. It can be shown that this estimate is unbiased and efficient.

The choice $n_1 = n_2 = n_3 = r$ constitute the so-called *standard measurement scheme* for qubits whose mean quadratic error matrix is

$$V_{3r}^{st}(\theta_S) = \frac{1}{r} \begin{bmatrix} 1-\theta_{S1}^2 & 0 & 0 \\ 0 & 1-\theta_{S2}^2 & 0 \\ 0 & 0 & 1-\theta_{S3}^2 \end{bmatrix}. \tag{2.8}$$

2.3 The measurement strategy

Indirect measurement means that the projective measurements are performed on the ancilla system (being in state θ_M) attached to the one we are interested in (θ_S). In the composite system (in state ρ_{S+M}) an indirect measurement corresponds to the observables of the form $I \otimes A_M$, where A_M is a self adjoint operator on the Hilbert space of system M. For the sake of simplicity, it is assumed, that A_M is a Pauli spin operator.

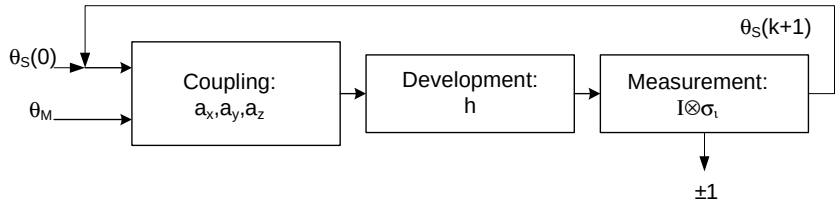

Figure 2.1: Signal flow diagram of indirect measurement

The above described *measurement strategy* is shown schematically in Fig. 2.1. At each time instant of the discrete time set, an ancilla qubit prepared in a known state is coupled to the unknown system S. They evolve according to the bipartite dynamics (2.6) for the sampling time h, and at the end of the sampling interval, a von Neumann measurement is performed on the ancilla qubit. At the next time instant, the previous steps are repeated.

The parameters of the strategy The coupling parameters a_1, a_2, a_3 of the Cartan decomposition (2.4-2.5) determine how (in terms of strength and direction) the measurement system is coupled to the unknown one. The sampling time h amplifies this effect and appears as a multiplicative factor to the coupling parameters. The state of the ancilla qubit (θ_M) can in principle be different at each time instant but we assume to use the same state for the ancilla all the times.

It has been shown in [15], that the above parameters of the strategy can be chosen such that a selective indirect scheme for estimating the second coordinate of the unknown qubit state vector θ_{S2}. The effect of the interaction parameters and that of the ancilla qubit state can then be collected in a constant $0 \leq c \leq 1$.

2.4 Mathematical problem statement

In this scenario let $x(k)$ be the second coordinate of the Bloch vector of the unknown qubit in the kth time step and c be a parameter that characterizes the state of the ancilla qubit. Then [15]

$$x(k+1) = \left\{ \begin{array}{l} \frac{x(k)+c}{1+cx(k)}, \text{with } \frac{1+cx(k)}{2} \text{ probability: } + \text{ measurement} \\ \\ \frac{x(k)-c}{1-cx(k)}, \text{with } \frac{1-cx(k)}{2} \text{ probability: - measurement} \end{array} \right\} \tag{2.9}$$

Note that if $c = 1$ then we obtain the direct standard measurement scheme, where

$$x(1) = \pm 1 \quad , \quad Prob(\pm 1) = \frac{1}{2}(1 \pm x(0)).$$

As shown in [14] all of the possible cases of states can be ordered in a line such a way, that after each measurement we jump in the neighboring state on the left or right side:

$$\cdots \rightleftarrows x_{-3} \rightleftarrows x_{-2} \rightleftarrows x_{-1} \rightleftarrows x_0 \rightleftarrows x_1 \rightleftarrows x_2 \rightleftarrows x_3 \rightleftarrows \ldots$$

where x_0=$x(0)$. Hence, if two outcome sequences contain the same number of + and - measurements, then the final state will be the same. With simple calculation it can be shown that their probabilities will be the same, too.

Let us denote the number of + and - measurements by ℓ_+ and ℓ_-, respectively, and $d = \ell_+ - \ell_-$. That is, the qubit will be in the state x_d after $\ell_+ + \ell_-$ time steps. If we denote $x_d = \frac{z_d}{q_d}$ with $\ell_+ > \ell_-$, then we can calculate the value of x_d with the following induction

$$(z_d, q_d) \rightarrow (z_d + cq_d, q_d + cz_d) = (z_{d+1}, q_{d+1}) \tag{2.10}$$

where $z_0 = x_0$ and $q_0 = 1$. It is easy to see that z_d and q_d are linear in x_0. Moreover if $z_d = a_d + b_d x_0$, then $q_d = b_d + a_d x_0$. Thus

$$x_d = \frac{a_d + b_d x_0}{b_d + a_d x_0} \tag{2.11}$$

Similarly we get:

$$x_{-d} = \frac{-a_d + b_d x_0}{b_d - a_d x_0} \tag{2.12}$$

where a_d and b_d are the same as in x_d. They can be calculated using the recursion, and they are polynomials of c.

It follows from the above properties that if we have the *measurement record*, that is, the observed measurement outcomes (+1s and -1s) and know the *parameter* c then we can calculate the state of the unknown qubit $x(k)$ $(= x_{\ell_+ - \ell_-})$ at any time instance from the initial state x_0. Therefore, **the only meaningful problem statement is to estimate the initial state x_0 from the measurement record and from c.**

Chapter 3

Conditional probability density function approaches

Traditionally, maximum-likelihood (ML) or Bayesian estimation procedures are very popular for quantum state estimation with compatible parametrization and distance [3]. Therefore, it is straightforward to use them in the indirect setting, too.

3.1 A conditional histogram approach

A simple method that is based on the direct estimation of the conditional probabilities in Eq. (2.9) is proposed in this subsection.

Let us assume to have a single copy of the unknown and ancilla qubit pair, and we have collected the outcomes of a projective measurement on the ancilla qubit $\{y(k)|k = 1, ..., N\}$. Let us fix the number of measurements N and the number of investigated systems states M, such that

$$k = 1, 2, ..., N; \quad N >> M$$

The systems starts from an unknown initial state x_0 to be estimated.

3.1.1 The estimation procedure

Data collection The first step of the estimation procedure is to collect the conditional relative frequencies ν_d of the "+1" measurement outcome to each considered relative systems states, i.e. to compute the pairs

$$(\nu_d, x_d), \quad d \in \{-M, ..., -1, 0, 1, ...M\} \tag{3.1}$$

This can be simply done by counting the number of the $+1$ measurement outcome when the system is in the state x_d and dividing it by the number of times when the system is in x_d.

Estimation The estimation is performed in three substeps.

(a) First we can construct an estimate for the relative states x_d by using Eq. (2.9)

$$\hat{x}_d = \frac{2\nu_d - 1}{c} \tag{3.2}$$

(b) Then we can use Eqs. (2.11) and (2.12) for $d \geq 0$ and $d < 0$, respectively to derive estimators for the unknown initial state x_0

$$\hat{x}_0^{(d)} = \frac{s \cdot a_d - b_d \hat{x}_d}{s \cdot a_d \hat{x}_d - b_d} = \frac{s \cdot a_d c - 2 b_d \nu_d - b_d}{2 s \cdot a_d \nu_d - s \cdot a_d - b_d c} \tag{3.3}$$

where $s = \mathrm{sign}(d)$.

(c) Finally, we can construct an overall estimate of x_0 from the above estimates $\hat{x}_0^{(d)}$ by averaging them, for example.

Properties of the estimator The estimators from (3.3) are biased, because they are nonlinear functions of the relative frequencies ν_d. The magnitude of the bias depends on the parameters of the problem, i.e. on x_0 and c.

Tuning the parameters of the algorithm The parameters of the algorithm can be chosen based on the parameters of the problem (x_0 and c), and that of the method d.

- *The number of measurements* N. The information is concentrated in the beginning of the trajectory, therefore we need to have it so large that we do not get too close to ± 1. A reasonable choice is when N is in the order of $1/c^2$.
- *The number of state points* M. It cannot be too large, but it should be large enough to get meaningful estimates. As x_d becomes more distant from x_0 the estimation (3.3) is less efficient. Therefore, x_d should not be too close to ± 1. A meaningful approach is to choose $M = \alpha \cdot \sqrt{N}$ with $0 < \alpha \leq 1$.
- The optimal way of combining the estimates $\hat{x}_0^{(d)}$ to form an overall estimate $\hat{x}_0^*$ is to be determined from statistical considerations. If d is close to 0 then (3.3) will have similar variance like (3.2). It is known that if we have independent unbiased measurements then the optimal linear measurement is if we have weights in ratio of reciprocal of variances. In this case the variance of $\hat{x}_0^{(d)}$ is constant times $\frac{1}{N_d}$, so the weights will be N_d in the weighted average.

3.1.2 Simulation results

The values $c = 0.1$, $x_0 = 0.4$ and $N = 1000$ were used in the simulations. Two different M values were investigated: $M = 10$ and $M = 20$.

If the estimator (3.2) resulted in a value with absolute value greater than 1, then $\hat{x}_0^{(d)} = \pm 1$ was used depending on the positivity of (3.2). The estimate was constructed from the individual estimators (3.3) by taking the weighted average.

In order to improve the efficiency of the estimate, we have repeated the simulation experiments using multiple copies of the same unknown and probe qubit pairs, and have averaged the estimates. The figures illustrating the simulation results depict the empirical mean values and the empirical mean square errors (MSEs) as a function of the used qubit pairs.

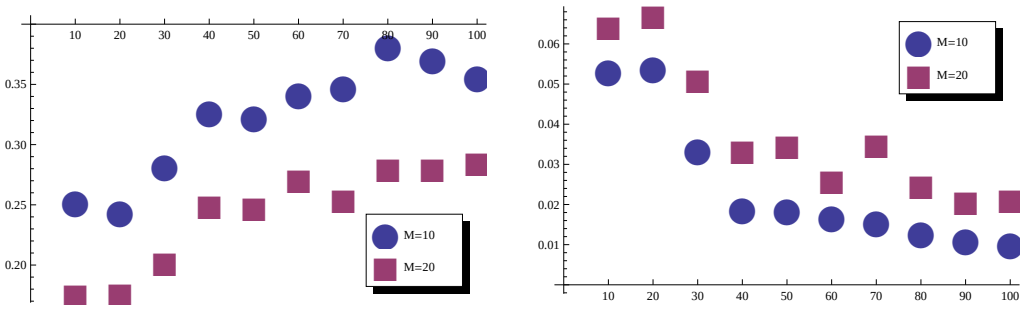

Figure 3.1: The mean and the MSE as a function of the number of used qubits

In the $M = 10$ case a large bias was observed because of the low number of relative frequency data, that is seen in the left sub-figure of Figure 3.1. For bigger numbers of the used qubit pairs this biasness vanishes, hence the mean square error becomes quite accurate as shown in the right-sub-figure.

In the $M = 20$ case one could expect to have better estimates but this is not the case. Because of the low number of relative frequency data the additional estimates that are far away form the initial state are so bad that they make the estimate even worse. In Figure 3.1 we can see that the convergence of the estimate to the true value is much slower, it is not surprising that mean square error is greater, too.

3.2 Bayesian state filtering

In this section a Bayesian parameter estimation method is proposed to estimate the initial state x_0. Let us suppose that $f_0(x)$ is an apriori density function of x_0. A sequence of indirect measurements is performed as before, and after each measurement the estimate of x_0 and the state evolution $x(k)$ are updated.

3.2.1 The Bayesian model of the state evolution

If we denote the outcome of the kth measurement by $y(k) \in \{-1, 1\}$, then we can calculate the actual state $x(k)$ recursively from Eq. (2.9) as

$$x(k) = \frac{x(k-1) + y(k)c}{1 + y(k) \cdot c \cdot x(k-1)}, \tag{3.4}$$

where $x(0) = x_0$. Of course, as we seen before, after each time step

$$x(k) = \frac{a_d x(0) + b_d}{a_d + b_d x(0)},$$

with some constant a_d and b_d that only depend on c and on the actual difference between the positive and negative measurement outcomes d. We will calculate the actual constants together with the simple recursion (3.4) in each step, because we update the probabilities also recursively.

Let the posterior probability distribution of the estimate x_0 be $f_k(x)$ after k step. Then the probabilities evaluate in the following way

$$\begin{aligned} f_{k+1}(x) &= Prob(x_0 = x | y(1), \ldots, y(k+1)) = \\ &= Prob(y(k+1) | x_0 = x, y(1), \ldots, y(k)) \cdot Prob(x_0 = x, y(1), \ldots, y(k)) \\ f_{k+1}(x) &\propto (1 + y(k+1) \cdot c \cdot x(k)) \cdot f_k(x) \end{aligned} \tag{3.5}$$

where $\propto$ means proportionality up to a normalizing constant.

3.2.2 Recursive estimation method

Let us fix the length of the measurement sequence to be N. If we do not know anything about the initial state, then the standard procedure is to use the uniform distribution on the state space as prior, i.e. let $f_0(x) = 1/2$. As the calculations are not analytically feasible, we only calculate the values of $f_k(x)$ on a grid of discrete values of $x_j \in [-1, 1]$.

Then, for each step (from $k = 0$ to $N - 1$) the following substeps are performed:

1. Perform a measurement and record $y(k + 1)$
2. Update the density function $f_{k+1}(x) = (1 + y(k + 1) \cdot c \cdot x(k)) \cdot f_k(x)$
3. Update the new state distribution $x(k + 1) = \frac{x(k)+y(k+1)\cdot c}{1+y(k+1)\cdot c\cdot x(k)}$

Having completed the above substeps, the whole sequence can be repeated on another qubit pair, etc. The obtained posterior $f_N(x)$ of the previous copy is used as the new apriori distribution on x_0 for the next copy of the unknown and probe qubit pairs:

$$f_0(x) := \frac{f_N(x)}{\int_{-1}^{1} f_N(x)dx}$$

3.2.3 Simulation results

The above described method was applied on multiple copies of qubit pairs with $x_0 = 0.4$ and $c = 0.1$. The length of each sequence was $N = 100$.

Figures 3.2 show the evolution of the probability density function of the estimate of x_0 after 10, 100 and 1000 steps. The convergence seems to be smooth, but the last sub-figure reveals a numerical issue that need to be solved, namely, we should somehow change the grid of calculation dynamically, to obtain accurate results.

From the probability distribution we can get easily a point estimation on x_0. The easiest way to calculate the mean of the density function. Another possibility to fit a normal distribution on the density function with least square method. With each method we will get similar results as a point estimator.

The empirical statistics of the estimate have been analyzed as functions of the used qubit pairs. The whole procedure was repeated 100 times and the results were averaged to calculate the empirical mean value and empirical mean square error (MSE). The empirical mean of the estimator can be shown in the left sub-figure of Figure 3.3, while the MSE is in the right sub-figure. We can see in Figure 3.3 that the estimate is biased, that is natural because we started from the uniform distribution on the $[-1, 1]$ interval. But it can also be seen that the convergence is fast, and the estimate is asymptotically unbiased. The right sub-figure of Figure 3.3 shows a convergence of MSE to zero in the order of $1/N$.

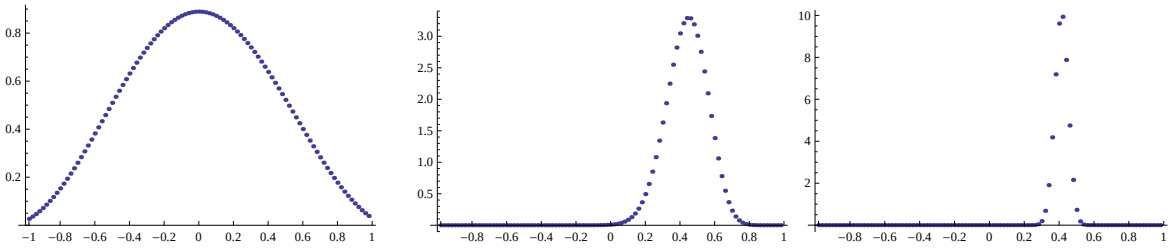

Figure 3.2: Density function after 10,100 and 1000 steps

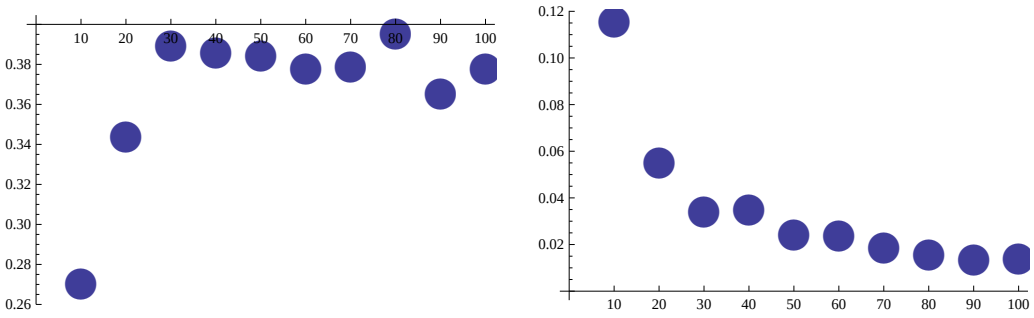

Figure 3.3: The mean and the MSE of the estimator as a function of the number of used qubits

Chapter 4

The martingale approach

The above two approaches result in a biased estimate of the initial state, therefore a novel estimation method based on the martingale property of the state evolution process is proposed in this section.

4.1 The martingale generated by the state evolution

The process described in (2.9) is a martingale, because it is easy to check that

$$E(x(k+1)) = x(k).$$

We can make use of this property if *we fix the values u, v ($u < x_0 < v$), and then we start the process from x_0 by performing indirect measurements until we reach either below u or above v. In that case we stop the process and note the final state.* In order to get a valid estimation method we assume that

- $|u| < 1$, $|v| < 1$
- x_0 is in the interval $[u, v]$.

Assume that we stop the procedure based on the above rules at time T, in that case T will be a stopping time, and according to Doob's optional stopping theorem [17]: $E(x(T)) = x_0$.

We can simplify the situation if we assume that the final state is exactly u or v, with probabilities p and q $(= 1 - p)$. In that case

$$E(x(T)) = p\ u + (1-p)\ v = x_0 \tag{4.1}$$

and one obtains after rearranging:

$$p = \frac{v - x_0}{v - u}$$

We can use Eq. (4.1) for state estimation if we replace the probabilities $(p, 1-p)$ with relative frequencies after n repeated settings using different unknown and ancilla qubit pairs. Let us denote $\nu_u = n_u/n, \nu_v = n_v/n$, where n_i $(i \in u, v)$ is the number of experiments with the final state i. Then

$$\hat{x}_0 = \nu_u\ u + \nu_v\ v = \nu_u\ u + (1 - \nu_u)\ v = v + (u - v)\nu_u$$

The variance of this estimate is:

$$Var(\hat{x}_0) = (u-v)^2\ Var(\nu_u) = (u-v)^2\ \frac{p(1-p)}{n}$$

because n_u has a binomial distribution with parameters (n, p). After substitution we obtain:

$$Var(\hat{x}_0) = \frac{1}{n}(v - x_0)(x_0 - u)$$

Note if we have $u = -1$ and $v = 1$ then we get an equivalent method to the direct standard method [11] with variance:

$$Var(\hat{x}_0) = \frac{1 - x_0^2}{n}$$

If u and v are in the interior of $[-1, 1]$ the we will get a smaller variance.

4.2 Stopping relative to the initial state

The above described procedure is not feasible because we do not know neither the state x_k nor x_0, so we can not know when we reach the stopping time. Therefore, an *initial state relative* method of detecting a stopping time is proposed where the final states are defined using a given distance from the unknown initial state.

4.2.1 Data collection

Let be $d(k)$ the number of $+1$ measurements minus the number of -1 measurements after the first k measurements. Let us repeat the measurements until $d(k)$ reaches $\pm D$, where D is a given integer, let be in this case $\tau = k$.

In this case τ is a stopping time, and the final state will be x_+ or x_-, accordingly if $d(\tau)$ is equal to $+D$ or $-D$. We can calculate x_+ and x_- from recursion (2.10) to have $x_+ = \frac{y_D}{q_D}$, where $y_D = a_D + b_D x_0$ and $q_D = b_D + a_D x_0$. Therefore

$$x_+ = \frac{a_D + b_D x_0}{b_D + a_D x_0}$$

Similarly we get:

$$x_- = \frac{-a_D + b_D x_0}{b_D - a_D x_0}$$

where a_D and b_D are the same as in x_+.

4.2.2 The estimator

We can use the above results to estimate the initial state x_0 using the $[x_-, x_+]$ interval.

The probability that the process ends up in x_+ is

$$p_+ = \frac{x_0 - x_-}{x_+ - x_-} = \frac{b_D + a_D x_0}{2b_D} = \frac{1}{2}\left(1 + \frac{a_D}{b_D}\ x_0\right) \tag{4.2}$$

that only depends on the ratio of a_D and b_D, and let $\gamma_D = \frac{b_D}{a_D}$. From the previous probability we can easily get an estimate for x_0. Denote ν_+ the relative frequency of the x_+ outcome, then

$$\hat{x}_0 = \gamma_D(2\nu_+ - 1) \tag{4.3}$$

Note that the estimate (4.3) is unbiased, because (4.2) is linear in x_0. Let us also remark that if we choose D big enough, then x_+ and x_- will be close to ± 1, so we use the whole trajectory and the estimation will be equivalent to the direct standard estimation method. As D goes to infinity γ_D decreasingly converges to 1, and so (4.3) converges to the direct standard estimator.

The variance of (4.3) is

$$Var(\hat{x}_0) = 4\gamma_D^2 Var(\nu_+) = \gamma_D^2 - x_0^2 \tag{4.4}$$

because ν_+ has a binomial distribution.

4.2.3 Possible generalizations

One of the key properties of the repeated weak quantum measurements is that the expectation value of the measurements is equal to the initial state. Therefore the martingale property holds for any other measurement strategy, too. Then the key element of using the martingale method is the knowledge of "stopping" probabilities. This is always a function of the initial state: $Prob = f(x_0)$, and from this we can easily construct an estimator: $\hat{x}_0 = f^{-1}(Prob)$.

This probability can be calculated from the terminating points, i.e. it can be calculated for almost all possible scenarios. The drawback is, that the variance is usually much bigger than the optimal one. Therefore, we need such a γ_D which is close to 1, that is we need to run the process quite long to achieve good variance.

4.3 Probability of returning to the initial state

At a stopping time the initial state is not completely destroyed, so we can try to get to the initial state again. That means we run the process further until we reach $d(k) = 0$.

Let us suppose that we are in x_+ and continue the measurements. Once again we can use the result of the previous section with initial state x_+ and with interval $[x_0, 1]$.

The probability that we return to x_0 is:

$$p_{return+} = \frac{1 - x_+}{1 - x_0} = \frac{b_D - a_D}{b_D + a_D x_0} = \frac{\frac{b_D}{a_D} - 1}{\frac{b_D}{a_D} + x_0}$$

Similarly the probability that we return to x_0 from x_- is

$$p_{return-} = \frac{1 - x_+}{1 - x_0} = \frac{b_D - a_D}{b_D - a_D x_0} = \frac{\frac{b_D}{a_D} - 1}{\frac{b_D}{a_D} - x_0}$$

So the probability we return to x_0 is

$$p_{return} = p_+ \cdot p_{return+} + (1 - p_+) \cdot p_{return-} = \frac{b_D - a_D}{b_D} = 1 - \frac{a_D}{b_D} = 1 - \frac{1}{\gamma_D}$$

This gives the natural $p_{return} = 1$, if $\gamma_D = \infty$, when the change is infinitesimal. Another example is $p_{return} = 0$, if $\gamma_D = 1$, when we have a direct Neumann-measurement as in the direct standard method.

4.3.1 Repeated measurements

Of course if the state returns to the initial state we can repeat the whole procedure to extract more information. The number of returns to initial state R will be geometrical distributed with parameter p_{return}. In the long run the average number of returns will be the mean value of R and

$$E(R) = \frac{1}{1 - p_{return}} = \gamma_D$$

This means that in average γ_D returns can be observed on one unknown and probe qubit pair. These numbers are independent therefore the approximated variance is:

$$\frac{Var(\hat{x}_0)}{E(R)} = \frac{\gamma_D^2 - x_0^2}{\gamma_D}$$

Note that this will be minimal if $\gamma_D = 1$.

4.4 Simulation investigations

The properties of estimator given in Eq. 4.3 were investigated by simulation, too. In the following let us assume that $c = 0.1$ and $x_0 = 0.4$. The simulations were performed for two different D values: first for $D = 10$ and then for $D = 100$. For these numbers we can easily calculate the a_D and b_D values that are $\gamma_{10} \approx 1.3106178$ and $\gamma_{100} \approx 1$. For each number of used qubit pairs we repeated the whole estimation procedure 100 times and calculated the empirical mean values and empirical mean square errors (MSEs).

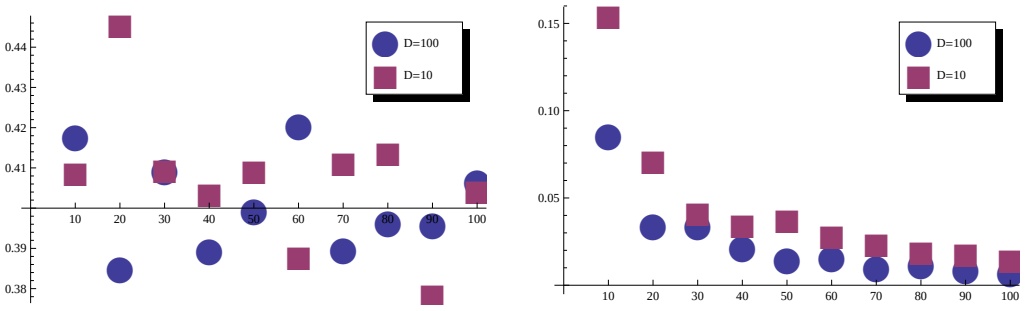

Figure 4.1: The mean and the MSE as a function of the number of used qubits

For $D = 10$ we observed that the mean is unbiased, the empirical values are close to the real initial state $x_0 = 0.4$ (left sub-figure of Figure 4.1). On the right sub-figure it is seen that the variance converges to the zero. The order of the convergence is $1/n$, where n is the number of used qubits.

For $D = 100$ the mean is also unbiased (Figure 4.1), and the variance is converging to zero with order of $1/n$, but a smaller coefficient is obtained than previously, which is in good agreement with (4.4). Note that this case is practically equivalent to the direct standard measurement, since $\gamma_{100} \approx 1$.

Chapter 5

Comparison and discussion

The three developed estimation method, the conditional histogram, the Bayes and the martingale approaches are compared in this section from the viewpoint of their efficiency. Thereafter the effect of the parameters in the measurements scheme, the effect of the unknown initial state x_0, and that of the ancilla qubit state c on the estimation quality have been investigated in the case of the martingale approach. Afterward a straightforward possibility of generalization is introduced to the multiple parameter case.

5.1 Efficiency

The efficiency of the indirect estimation methods is evaluated by comparing the mean square error (MSE) of their estimates with the theoretical variance of the direct standard estimation method

$$\eta = \frac{MSE_{method}}{Var_{dstandard}} \tag{5.1}$$

where $Var_{dstandard} = \frac{1-x_0^2}{n}$ with n being the used qubit pairs.

The efficiencies of the previously detailed methods can be seen on the Figure 5.1.

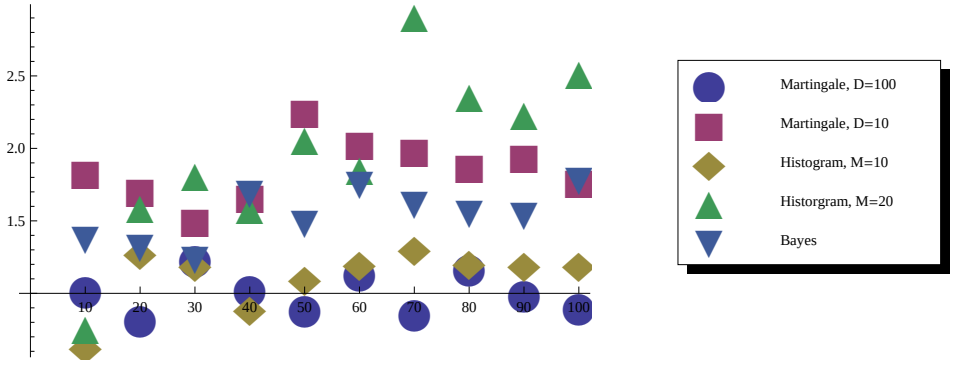

Figure 5.1: The efficiency of different methods

Conditional histogram approach On Figure 5.1 M=10 case is represented by diamonds and M=20 case by upper triangle. This approach provides biased estimates, therefore its relatively good efficiency shown should be taken with reservation. If the parametrization is good (M=10), we can achieve quite efficient estimation, although this seem to be a tough problem without any information on the initial state.

Bayes approach On Figure 5.1 this is represented by lower triangle. The estimate in this case is also biased but the efficiency is not very good either. The method does not need any parametrical setting, which can ruin the estimation. Nevertheless the computation demand is high and it only produce mediocre efficiency.

Martingale approach On Figure 5.1 D=10 case is represented by square, D=100 case is represented by circle. As we have seen before, the estimate based on the martingale approach is unbiased, and its efficiency with D=100 is close to the optimal value 1. The optimal choice of D is crucial, but it is enough to choose D big enough, but the actual value depends on parameter c.

The above discussion show that the martingale approach gives the best quality estimate from the statistical viewpoint, that is unbiased and efficient at the same time. In addition, we can easily tune the parameter D.

5.2 The effect of the measurement scheme parameters on the estimation quality

We have already seen, that the best quality result has been obtained using the martingale approach with big D ($D = 100$). Therefore, the effect of the parameters in the measurement scheme has been investigated only in this case.

The effect of the initial state x_0 The martingale approach has been investigated by using different initial states $x_0 = 0.1; 0.4; 0.9$ but with the same parameter value $c = 0.1$. The simulation results are shown in Figure 5.2 where the obtained values are denoted by a full diamond for $x_0 = 0.1$, a circle for 0.4, and a square for 0.9.

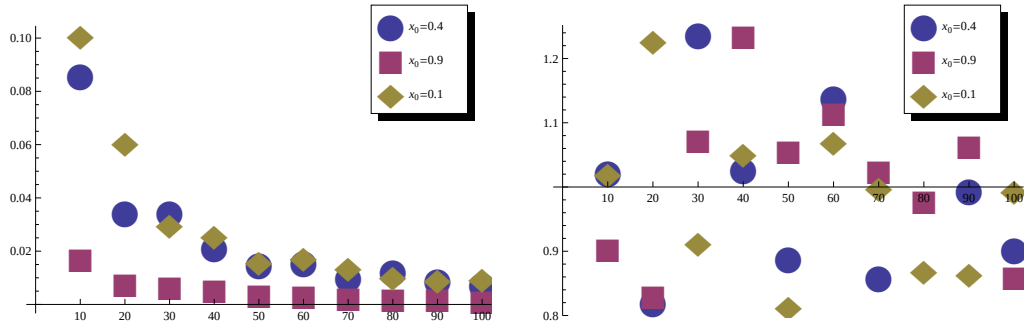

Figure 5.2: Comparison of MSE and efficiency for different x_0 values (0.1: diamond, 0.4: circle, 0.9: square)

The left sub-figure of Figure 5.2 shows the variation of the MSE with the number of used qubit pairs. The graphs show a similar shape with an initial state dependent scaling coefficient. It is not surprising because we expect from (4.4) that the coefficient is $1 - x_0^2$ (because $\gamma_{100} \approx 1$). If we plot the efficiency η from (5.1), we can see (right sub-figure of Figure 5.2) that efficiency is close to 1, there is no trivial dependence on the value of x_0.

The effect of coupling parameter c The martingale approach with $D = 100$ was used here with different values of parameter c but with the same initial state value $x_0 = 0.4$. The simulation results are shown in Figure 5.2 where the obtained values are denoted by a full circle for $x_0 = 0.1$, a square for 0.2, and a diamond for 0.05. Note that one needs more measurements until the stopping points for $c = 0.05$ compared to the standard $c = 0.1$ case, and one has less with $c = 0.2$.

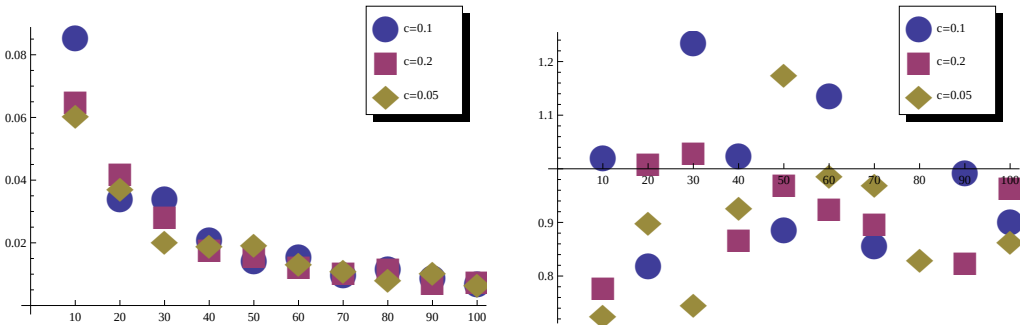

Figure 5.3: Comparison of MSE and efficiency for different c values (0.05: diamond, 0.1: circle, 0.2: square)

It can be seen in both sub-figures of Figure 5.3 that neither the MSE nor the efficiency does not depend on the value fo c in the investigated parameter domain. This is a consequence of the fact that the same $x_0 = 0.4$ was used, and the γ_{100} is very close to 1 for these c parameter values with $D = 100$. With further decreased c a greater value of D should be chose to achieve the same efficiency. In conclusion: a greater value of c does not affect negatively on the quality of the estimate but a smaller number of measurements is needed in this case.

5.3 Generalization to the multiple parameter case

Until now the single parameter simple case described in 2.3 was used to estimate selectively one of the components of the unknown qubit's Bloch vector, similarly to the so called standard measurement scheme [12] for single qubits. A straightforward modification of the measurement setup, that is, the change of the interaction direction and the observable together with the initial state of the probe qubit, leads to the estimators of the other two Bloch vector components.

Generalization to n**-level quantum systems** Consider a k-level quantum system with density matrix ρ acting on the Hilbert space $\mathcal{H}$. Then the state of n identically prepared

quantum systems is described by $\rho_n := \rho^{n\otimes}$ acting on the n-fold tensor product Hilbert space $\mathcal{H}_n$. When $\dim\mathcal{H} = k$, we can identify the operators of $\mathcal{H}_n$ with matrices of $k^n \times k^n$. Since the density matrices are self-adjoint matrices with unit trace, they can be characterized by $m = k^2 - 1$ real parameters forming the parameter vector θ, the generalized Bloch vector.

Denote by E_{ij} the $k \times k$ matrix units and set

$$\begin{aligned} Z_{ii} &:= E_{ii} && (1 \le i < k), \\ X_{ij} &:= E_{ij} + E_{ji} && (i < j), \\ Y_{ij} &:= -\mathrm{i}E_{ij} + \mathrm{i}E_{ji} && (i < j). \end{aligned}$$

The spectrum of Z_{ii} is $\{0, 1\}$ and the spectrum of X_{ij} and Y_{ij} is $\{-1, 0, 1\}$. These observables can be used to estimate the $k^2 - 1$ real parameter of the $k \times k$ density matrix selectively. If each observables are measured r times, then $R = r(k^2 - 1)$ copies of the quantum system are used.

Using the above state parametrization and observables, one can construct estimators that can be applied in the indirect setting similarly to the ones described in the qubit case above.

Chapter 6

Conclusions

The simplest possible discrete time indirect measurement scheme has been investigated here, where both the unknown and the measurement quantum systems are quantum bits. The measurements applied on the measurement qubit were chosen to be the classical von Neumann measurements using the Pauli matrices as observables. The repeated measurements performed on the measurement subsystem of the composite system were used to construct estimators of the initial state of the unknown system.

Based on the statistical properties of the considered indirect measurement scheme [15], three related but different approaches have been proposed and investigated:

- a direct estimation procedure that is based on the estimated relative frequencies of the characterizing conditional probability densities,
- a Bayesian recursive approach for state estimation,
- a martingale approach that bases the estimator on the stopping times of the state evolution as a martingale driven by the repeated measurements.

The unbiasedness and the mean square error of the estimate has been analyzed by analytical computations and computer simulation. It has been shown that the martingale approach gives an unbiased estimate, while the other two approaches result in an at most asymptotically unbiased estimate.

It has been shown by simulation that the martingale approach gives the best quality estimate from the statistical viewpoint, that is unbiased, and efficient at the same time. The other methods do not perform much worse result, but there are some numerical settings which are harder to fulfill.

The effect of the parameters of the estimation scheme on the quality of the estimate has also been investigated by simulation, and possible generalization are also mentioned.

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