Constraint $\mathcal{H}_\infty$ control for discrete-time LPV systems by interpolating among linear feedback gains

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Abstract

This paper proposes an interpolation based control framework as a possible solution to the constrained $\mathcal{H}_\infty$ control problem of discrete-time linear parameter varying (LPV) systems. The control policy is constructed by interpolating among pre-designed, unconstrained state feedback controllers. The required control performance and the fulfillment of the hard constraints are assured. Moreover the domain of applicability of the proposed controller is significantly larger, than of a single state feedback policy.

1 Introduction

In the last years the increased computational power provided by the continuously developing computer architectures made it possible to design and implement control methods, which are able to take hard constraints into consideration [3],[14].

In this paper the constrained $\mathcal{H}_\infty$ control is considered. The aim of the control synthesis is to minimize the induced $\mathcal{L}_2$ gain between the generalized deterministic disturbance input (w) and the performance output (z). The state x of the system and the control input $u(x)$ are subject to hard constraints $x \in X, u(x) \in U$ with apriori chosen sets of the signals X and U. The state feedback control policy $u(x)$ solves the constrained $\mathcal{H}_\infty$ problem if either of the following (equivalent) conditions are fulfilled:

(C1) $u(x)$ is defined so that $x \in X, u(x) \in U$ and there exists positive definite storage function $V(\cdot) > 0$ satisfying the dissipation inequality $z^T z - \gamma^2 w^T w + V(x_+) \leq V(x)$

(C2) $u(x) = \arg\min_{u\in U} \max_{w\in W} \sum_{k=1}^{N-1} z_k^T z_k - \gamma^2 w_k^T w_k + V(x_{k+N})$ where $V(\cdot) > 0$ is a storage function of the unconstrained $\mathcal{H}_\infty$ problem or satisfies the more general conditions given in [8].

The first one is the passivity based condition, while the second one considers the $\mathcal{H}_\infty$ problem as a special case of the zero sum differential games [8]. Depending on which condition is employed different formulations of the solution can be derived.

A receding horizon control (RHC) structure is obtained by applying (C2). The min-max optimization is performed at each time instant in a predictive manner; always with the fresh data and only the first control input is applied [6], [15], [8]. For nonlinear systems [9] proves (using (C1)) a necessary and sufficient condition for the solvability of the constrained $\mathcal{H}_\infty$ problem if Euclidean norm input constraints are present. In this case, the controller can be given in a closed, nonlinear state feedback form. Unfortunately, to satisfy the given conditions, one has to solve nonlinear matrix inequalities. Generic tools for the solution does not exist.

Efficient solutions exist only for linear time invariant systems ([2]). In linear case, the RHC controller can be implemented by decreasing the complexity of the on-line optimization (see e.g. [11]). Furthermore, [7] proved that the min-max optimization can be rewritten to an equivalent convex optimization problem by QP (quadratic program), SDP (second order cone program) or SDP (semi-definite program) methods. For linear time invariant system, an explicit solution is available if the control policy is linear as well [1]. Dynamic programming provides an explicit solution for $u(x)$ of the problem (C2) given in a piecewise affine form defined over a polyhedral subset of the states ([17]). These results can hardly be extended to nonlinear systems.

The class of linear parameter-varying (LPV) systems is between the nonlinear and linear systems. They are able to model nonlinear dynamics while preserving many useful properties of linear systems. This is a reason why the LPV modelling is used widely in control design ([16],[13]). Also in constraint $\mathcal{H}_\infty$ control, the LPV structure gives a chance to extend the efficient algorithms developed for LTI systems to the nonlinear case. In [5] a (C1) type structure is proposed by a model predictive $\mathcal{H}_\infty$ controller for LPV systems. $V(x)$ has been chosen to be quadratic, the feedback is supposed to be a linear state feedback one, $u(x) = Kx$. Thus, (C1) could be casted to linear matrix inequality (LMI), to a convex optimization problem. The constraint handling (driven back to LMI) is a very conservative estimation of the robust invariant set. The LMI-s generated by the constraints can be satisfied only for states that are very close to the origin. This conservativeness highly limits the applicability of the method. [4] suggests an application oriented form of the LMI $\mathcal{H}_\infty$ based solution for parameter varying systems.

There is a room to improve constrained $\mathcal{H}_\infty$ control methods, because existing methods for parameter dependent systems may be time consuming (RHC), difficult to solve (non-convex optimization, nonlinear matrix inequalities), too conservative (LMI based solutions). On the other hand, [20] and [24] show the computational of a valuable measure of the constrained $\mathcal{H}_\infty$ controllers, the robust invariant sets for LPV systems.

To overcome the difficultly tractable aspects of the mentioned methods, the paper offers an interpolation based control design for LPV systems. The motivation of the present work is to eliminate the drawbacks of the existing methods.

- Our method interpolates among linear feedback gains $u(x) = Kx$. These controllers can be synthetized apriori the application, therefore no online computation is required.

- Off-line computation of the region of applicability of the controllers
- The only real-time computational demand is the linear interpolation among the pre-computed controllers with performance upgrade if needed.

The paper is organized as follows. In the next section the problem itself is formulated and the outline of the solution is presented. In section 3 two algorithms are presented for the approximate construction of the maximal and minimal disturbance invariant sets. Both are needed in the determination of the region of applicability of the controller. The interpolation based control schema is presented in section 3 and 4 and its main properties are also derived there. The algorithm is tested by simulation on the control of a simple LPV system. In section 6 the results are summarized and the main conclusions are drawn.

2 Problem formulation

Let a discrete-time, linear parameter-varying (LPV) system be given in polytopic form by

$$\begin{aligned} x_+ &= A(\delta)x + B_1 w + B_2(\delta)u \\ z &= C(\delta)x + D_1(\delta)w + D_2(\delta)u \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $z \in \mathbb{R}^{n_z}$ and $w \in \mathbb{R}^{n_w}$ are the states, control input, performance output and the disturbance signals respectively. (x_+ is the successor state at a given moment k). The polytopic representation of the model is written as

$$\delta \in \Delta, \;\; \Delta = \{\delta = [\delta^1, \ldots, \delta^L] \mid \delta^i \in \mathbb{R}^+, \textstyle\sum_{i=1}^L \delta^i = 1\} \tag{2}$$

$$[A(\delta), B_2(\delta), C(\delta), D_1(\delta), D_2(\delta)] = \textstyle\sum_{i=1}^L \delta^i \cdot [A_i, B_{2i}, C_i, D_{1i}, D_{2i}] \tag{3}$$

with matrices known $A_i, B_{2i}, C_i, D_{1i}, D_{2i}$ of appropriate dimension. It is assumed that the state x_k and the parameter vector δ_k are measured at each time instant k and – for simplicity – B_1 is constant. Our aim is to find a state feedback controller solving the constrained $\mathcal{H}_\infty$ problem defined as follows:

Problem 1 (constrained $\mathcal{H}_\infty$) Let $0 < \gamma^* < 1$ be a given predefined performance level. Find a stabilizing state feedback control policy $u = u(x)$ for system (1) so that

(i) there exists a positive definite quadratic storage function $V : \mathbb{R}^{n_x} \to \mathbb{R}^+, V(x) = x^T P x$ with a symmetric and parameter independent solution matrix $P \in \mathbb{R}^{n_x \times n_x}, P > 0$ and a positive scalar $\gamma < \gamma^*$ s.t. the state x and output z of the closed-loop system $x_+ = A(\delta)x + B_2(\delta)u(x) + B_1 w$, $z = C(\delta)x + D_2(\delta)u(x) + D_1(\delta)w$ satisfies the dissipation inequality

$$z^T z - \gamma^2 w^T w + V(x_+) \leq V(x) \tag{4}$$

for all possible disturbance sequence $[w_0, w_1, \ldots] : \sum_{i=0}^\infty w_i^T w_i < \infty$. Moreover

(ii) there exists a nonempty set $S \subset \mathbb{R}^{n_x}$ so that $x \in S$ implies

$$\begin{gathered} x_+ = A(\delta)x + B_2(\delta)u(x) + B_1 w \in S \\ x \in X, \;\; u(x) \in U \\ \forall w \in W, \; \forall \delta \in \Delta \end{gathered} \tag{5}$$

where (5) are the prescribed state/input constraints and $X \subset \mathbb{R}^{n_x}, U \subset \mathbb{R}^{n_u}, W \subset \mathbb{R}^{n_w}$ are convex polytopes containing the origin in theirs interior.

The Part (i) is the generic formulation of the the unconstrained $\mathcal{H}_\infty$ control problem[1]. Satisfying (4), it is equivalent to assure the induced $\mathcal{L}_2$ norm of the transfer from all possible disturbances w to the performance output z is less than γ (respectively γ^*). Applying the Part (ii) $u(x)$ solves the constrained $\mathcal{H}_\infty$ control problem, i.e. beside providing the requested performance the control input and state satisfies the prescribed constraints $u \in U, x \in X$ if the disturbance comes from the convex set W. Based on U, X and W it is

[1] The controllers fulfilling the conditions given in the Problem 1 are called - analogously to the linear case - $\mathcal{H}_\infty$ controllers, even though the system is nonlinear and only induced $\mathcal{L}_2$ operator norm can be defined over.

possible to construct S, a disturbance invariant (d-invariant sets, [12]) set of the closed loop dynamics. By means of invariance if the initial state x_0 lies inside S then the trajectory of the closed loop system never leaves S, even in the presence of exogenous disturbance $w \in W$. Moreover, $x \in S$ implies $u(x) \in U$ and $x \in X$. Therefore S can be considered as a *region of applicability* of the controller $u(x)$.

The control policy $u(x)$, solving the problem defined above, is constructed by interpolating among alternate linear and constant state feedback policies $u = Kx$.

The choice of a linear feedback may be too conservative with respect to nonlinear policies[2]. The reason of applying constant feedback gain resides in the fact that the computation of K and the approximation of the d-invariant set is less difficult than in the case of a nonlinear feedback. If $V(x)$ is a quadratic function in the form of $x^T Px$, then it is easy to find an appropriate solution to the Part (i) of Problem (1). In this case the gain of the stabilizing $\mathcal{H}_\infty$ controller K can be computed by replacing Kx into (1) to close the loop. Since, the controller has to fulfil the robust dissipative condition in (4), the inequality can easily be rewritten as a linear matrix inequality. The problem can therefore be solved efficiently for variables γ and P by using convex programming (For the details see the Appendix A.1.).

If γ is a free variable to assign and one only wants to assure that $\gamma \leq \gamma^*$, different linear K_i state feedback controller can be designed. Hence, alternate constrained $\mathcal{H}_\infty$ controller assuring different disturbance attenuation levels can be synthesized. To compare the possible controllers it is not sufficient to rank them with respect to the achieved minimal performance, γ_i. The feedback gain of those controllers providing low disturbance level is usually high. Consequently, large control actions are taken with low γ_is.

A valuable comparative measure of the $\mathcal{H}_\infty$ controllers may be the d-invariant set itself. It reflects the possible amount of applicable states in the closed loop. The relation between the $\mathcal{L}_2$ gain achieved and the robust invariant set generated by the controller $u(x) = Kx$ are strictly attached. In case of input constraints, controllers inducing small $\mathcal{L}_2$ gain have in general small robust invariant sets due to the high feedback gain. Consequently the *best* controllers can only be used in the small neighborhood of the origin. This fact restricts the applicability of the robust constrained controller nearby the origin over a small invariant set of the closed loop states. Therefore, minimizing γ in the design process sometimes leads to almost empty robust invariant sets, respectively only unconstrained solutions are given.

In order to overcome this trade-off, i.e. to enlarge the region of applicability of the constrained $\mathcal{H}_\infty$ control while keeping the performance under a predefined level (γ^*), a novel interpolation based control scheme is proposed. First, a set of unconstrained $\mathcal{H}_\infty$ controllers (solving only Part (i) of Problem (1) is designed with performance $\gamma \leq \gamma^*$. This set is then augmented with an *auxiliary* controller that may have bad performance ($\gamma \gg \gamma^*$), but what is more important has large d-invariant set. It can be shown that the interpolation among the appropriately chosen controllers assures better performance attenuation than a single mode control law. The performance of the control remains under the predefined level (the $\mathcal{L}_2$ gain remains under γ^*). Moreover, the d-invariant set of the closed-loop system is determined by the maximal d-invariant set of the 'auxiliary' controller providing a large set of the states. In this sense the region of applicability of the $\mathcal{H}_\infty$ control can be enlarged.

Furthermore, the linear interpolation among the pre-designed state feedback controllers gives an additional freedom compared to the application of a single linear $\mathcal{H}_\infty$ control rule all over the time.

3 Disturbance invariant sets for polytopic systems

The section summarizes the necessary definitions and algorithms for the computation of the robust disturbance invariant sets.

As it was previously mentioned, the d-invariant sets of the closed-loop system play a key role in the determination of the region of applicability of the applied control policy. At the same time the exact computation of these sets is generally a complex problem, even if the system is linear time invariant. In the literature several papers can be found discussing the properties of the d-invariant sets and proposing practical approximations for them. One of the first and basic work is [12], where the definition of d-invariance and some elementary properties of the d-invariant sets of linear time invariant systems are clarified. For the approximation of these sets one can refer to [18],[21],[23] or [22]. [19] proposes efficient methods for the construction of the maximal invariant set of a polytopic system, but it focuses only on the disturbance free

[2] Different state dependent laws may be applied such as nonlinear [?] or in the unconstrained case a parameter dependent [?].

case. Revising and extending these earlier results this section proposes simple algorithms for the approximations of the d-invariant sets of system (1), generated by a parameter-independent state feedback controller $u = Kx$. Note, the closed loop system remains parameter varying.

First of all the definition of the d-invariance has to be recalled:

Definition 1. (d-invarinat set, minimal, maximal) The set $S \subseteq X$ is a d-invariant set generated by the controller K if S is the d-invariance set of the closed loop system $(A(\delta) + B_2(\delta)K)x + B_1 w$, i.e. $x \in S$ implies $Kx \in U$ and $(A(\delta) + B_2(\delta)K)x + B_1 w \in S$ for all $w \in W$. The smallest and largest d-invariant sets are called the minimal and maximal d-invariant sets and are denoted by $\underline{S}$ and $\overline{S}$, respectively. By definition each d-invariant set S satisfies the relations $\underline{S} \subseteq S$ and $S \subseteq \overline{S}$.

For our interpolation based control algorithm both the minimal and the maximal d-invariant sets is needed. Therefore, two algorithms are proposed for the approximate construction of the maximal and the minimal d-invariance set.

3.1 Outer approximation of the maximal d-invariant set

The invariant set algorithm proposed by [19] for the disturbance-free case can be used with slight modifications to determine the maximal d-invariant set. For this, assume that the constraints $x \in X$ and $Kx \in U$ are expressed by one linear constraint set $A_S x \le b_S$, where each row of $[A_S, b_S]$ corresponds to a linear constraint of the form $a^T x \le b$. Consider now the the sequence of sets $S_0, S_1, S_2, \ldots$ defined as follows: $S_0 = \{x \,|\, A_S x \le b_S\}$, $S_t = \{x \in S_{t-1} \,|\, (A(\delta) + B(\delta)K)x \in S_{t-1},\ \forall w \in W, \delta \in \Delta\}$, $t > 0$. If S_0 is a polytope then S_t and $S_i \subseteq S_{i-1}$ are polytopes as well. The following lemma proves the convergence of the approximation towards $\overline{S}$.

Lemma 1. (convergence $\overline{S}$) *If the sequence of sets* $S_0 = \{x \,|\, A_S x \le b_S\}$, $S_t = \{x \in S_{t-1} \,|\, (A(\delta) + B(\delta)K)x \in S_{t-1},\ \forall w \in W, \delta \in \Delta\}$ *is convergent then* $\lim_{t\to\infty} S_t = \overline{S}$

Proof. (The proof is similar to the methods in [19].) It is proved that $S^* = \lim_{t\to\infty} S_t$ is a d-invariant set and it contains $\overline{S}$. Since $S_t \subseteq S_{t-1}\ \forall t$ and $S^* = \lim S_t$ thus the set $S' = \{x \in S^* \mid (A(\rho) + B_2(\rho))x + B_1 w \in S^*,\ \forall w \in W, \forall \delta \in \Delta\}$ can not be a real subset of S^*. Consequently $S' = S^*$, i.e. S^* is d-invariant. Suppose now $\overline{S} \nsubseteq S^*$. This means that there exists j s.t. $\overline{S} \subseteq S_{j-1}$, but $\overline{S} \nsubseteq S_j$. Since $\overline{S}$ is d-invariant, for all $x \in \overline{S}$ $(A(\rho) + B_2(\rho))x + B_1 w \in S_{j-1}$. This implies that $\overline{S} \subseteq S_j$, which contradicts the assumption. ■

The algorithm constructing the outer approximation of the maximal d-invariant set can be given as follows:

Algorithm 1. (Outer approximation of the maximal d-invariant set)

1. *Initialize* $c_{max} = 0$, $S_0 = \{x | A_{S_0} x \le b_{S_0}\}$, $A_{S_0} = A_S, b_{S_0} = b_S$, $t = 1$
2. *Set* $M = [], m = []$
3. *Perform the following steps while j is not larger than the number of rows in* $A_{S_{t-1}}$
   (a) *Take the jth inequality* $a^T x \le b$
   (b) *Check whether there exists* $x \in S_{t-1}$,$w \in W$ *and* δ *s.t.* $(A(\delta) + B(\delta)K)x + B_1 w \notin S_{t-1}$. *For this, compute for each pair* (A_i, B_{2i}), $i = 1 \ldots L$

$$c_i = \max_{w\in W, x\in S_{t-1}} a^T[(A_i + B_{2i}K)x + B_1 w] - b \tag{6}$$

$$(x_i^*, w_i^*) = \arg \max_{w\in W,\ x\in S_{t-1}} a^T[(A_i + B_{2i}K)x + B_1 w] - b \tag{7}$$

If for any i $c_i > 0$ save the inequality $a^T(A_i + B_{2i}K)x \le b - a^T B_1 w_i^*$, *i.e. let*

$$M = \begin{bmatrix} M \\ a^T(A_i + B_{2i}K) \end{bmatrix} \qquad m = \begin{bmatrix} m \\ b - a^T B_1 w_i^* \end{bmatrix} \tag{8}$$

   (c) *let* $c_{\max} = \max(c_{\max}, c_1, \ldots, c_L)$

4. *let* $A_{S_t} = \begin{bmatrix} A_{S_{t-1}} \\ M \end{bmatrix}$, $b_{S_t} = \begin{bmatrix} b_{S_{t-1}} \\ m \end{bmatrix}$ *and* $S_t = \{x \,|\, A_{S_t}x \leq b_{S_t}\}$.

5. $[A_{S_t}, b_{S_t}] = \mathrm{reduce}(A_{S_t}, b_{S_t})$

6. *if* $c_{\max} < \epsilon$ *then let* $\hat{\overline{S}} = S_t$ *and stop, else* $t := t+1$ *and go to step 1*

The algorithm assumes first, that the set $\{x \,|\, A_S x \leq b_S\}$ generated from the constraints is equal to a d-invariant set. Afterwards, by going through each constraint it checks whether there exist an $x \in S$ that violates the constraint, i.e. whether there exists an $x \in S$ and w s.t. $(A(\delta) + B_2(\delta)K)x + B_1 w \notin S$. In this case a new constraint is added to the existing set of rules. Involving more and more constraints, an outer approximation is given for the maximal d-invariant set. By increasing the number of the rules, the invariant set becomes smaller and smaller till it covers the maximal set with the predefined precision ϵ.

Since the algorithm only adds and never removes constraints it is worth revising occasionally the constraint set and removing the redundant constraints. This can be performed in a straightforward way by linear programming. The details can be found in [19]. The constraint set reduction is indicated in the algorithm above by calling the *reduce()* function in step 5.

The presented algorithm can not construct the maximal d-invariant set in finite steps. Therefore the procedure is completed with a terminal condition $c_{max} < \epsilon$, where the variable $c_{\max}$ measures the 'difference' between two consecutive sets and if it is acceptable small the computation stops.

The execution steps of the algorithm can be followed in Figure 1, where the calculation of the maximal d-invariant set $\overline{S}_b$ of controller K_b (defined in section 6) is presented. It can be seen that the algorithm converges after 5 steps.

3.2 Inner approximation of the minimal d-invariant set

In linear time invariant case the minimal d-invariant set is the set that contains all trajectories of the closed-loop system starting from $x_0 = 0$ ([12]). This can be extended for the LPV case. Therefore, let us consider the sequence of sets $S_0, S_1, \dots$ defined as follows $S_0 = \emptyset$, $S_t = S_{t-1} \cup \{(A(\delta) + B(\delta)K)x + B_1 w \,|\, x \in S_{t-1}, w \in W, \delta \in \Delta\}$. It is obvious that $S_{t-1} \subset S_t$. The following lemma proves that if this sequence is convergent then it converges to $\underline{S}$.

Lemma 2. (convergence $\underline{S}$) *If the sequence of sets* $S_0 = \emptyset$, $S_t = S_{t-1} \cup \{(A(\delta) + B(\delta)K)x + B_1 w \,|\, x \in S_{t-1}, w \in W, \delta \in \Delta\}$ *is convergent* [3] *then* $\lim_{t\to\infty} S_t = \underline{S}$.

Proof. (The proof is based on [19].) It is proved that $S^* = \lim_{t\to\infty} S_t$ is d-invariant and is contained in $\underline{S}$. Since $S_{t-1} \subseteq S_t$ and $S^* = \lim_{t\to\infty} S_t$ then the set S^* can not be a real subset of $S' = S^* \cup \{(A(\delta) + B(\delta)K)x + B_1 w \,|\, x \in S^*, w \in W, \delta \in \Delta\}$, i.e. $S' \equiv S^*$. Consequently S^* is d-invariant. Since $0 \in W$ and $\underline{S}$ is d-invariant, $x \in \underline{S}$ implies $\left(\prod_{i=1}^{k} A(\delta_i) + B_2(\delta_k)K\right) x \in \underline{S}$. Since $A(\delta) + B_2(\delta)K$ is quadratically stable, $\left(\prod_{i=1}^{k} A(\delta_i) + B_2(\delta_i)K\right) x \to 0$ as $k \to \infty$. Since $\underline{S}$ is closed thus $0 \in \underline{S}$. By the d-invariance of $\underline{S}$, all trajectories starting from $x_0 = 0$ run inside $\underline{S}$. Consequently $S_t \subset \underline{S}$ and thus $S^* \subseteq \underline{S}$. ■

The algorithm computing the inner approximation of the minimal d-invariant set can be given by:

Algorithm 2. (Inner approximation of the minimal d-invariant set)
Let M_X *denote a matrix containing in its columns the corner points of an arbitrary convex set* X *and let* $\mathrm{col}(M_X)$ *be the number of columns in* M_X.

1. *Initialize* $M_{S_0} = []$, $t := 1$

2. *Set* $M = []$ *and perform the following step for all triplets* (i, j, k), *where* $i \in \{1 \dots L\}$, $j \in \{1, \dots, \mathrm{col}(M_{S_{t-1}})\}$, $k \in \{1 \dots \mathrm{col}(M_W)\}$.

$$M := \begin{bmatrix} M & \{(A_i + B_{2i}K)x_j + B_1 w_k\} \end{bmatrix} \tag{9}$$

[3] Although the convergence can also be proved by applying the same reasoning as [12] in section 4, but being not relevant only this weaker lemma is proved now.

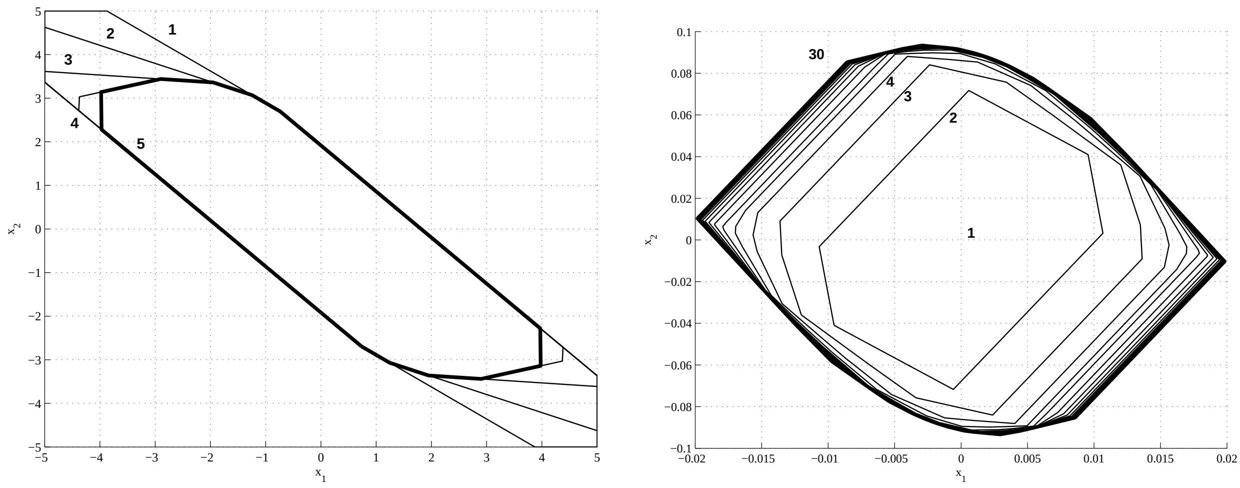

Fig. 1: Construction of the maximal d-invariant set $\overline{S}_b$ (left) and the minimal d-invariant set $\underline{S}_a$ (right)
.

3. $S_t = \text{convhull}(\begin{bmatrix} M & M_{S_{t-1}} \end{bmatrix})$

4. if $\text{diff}(S_t, S_{t-1}) \leq \epsilon$ *then* $\hat{\underline{S}} = S_t$ *and stop, else let* $t = t+1$ *and go to step 2.*

The algorithm computes iteratively the corner points of the sets S_t. Since the minimal d-invariant set can not be constructed in finite steps, an appropriate terminal condition is inserted again.

Figure 1 demonstrates the computation of the minimal d-invariant set of controller K_a (defined in section 6). The algorithm starts from the set containing only the origin (step 1) and after 30 steps it reaches the minimal d-invariant set $\underline{S}_a$.

4 Interpolation based controller

In this section the interpolation based control structure is presented and its main properties are derived. The control structure is similar to the controller proposed by [24], [20] for uncertain linear systems.

Suppose m different unconstrained $\mathcal{H}_\infty$ controllers have already been designed for the system (1). Let they be given by the following ordered pairs:

$$(K_1, \gamma_1), (K_2, \gamma_2), \ldots, (K_m, \gamma_m) \tag{10}$$

where γ_i is the induced $\mathcal{L}_2$ gain provided by the state feedback control $u = K_i x$. The ordering is according to γ, i.e. $0 < \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{m-1} \leq \gamma^* \ll \gamma_m < \infty$. (Note that the last controller can not be used in itself since the performance it provides is worse than the acceptable level.) Consider now the extended system constructed from the m closed loop dynamics formed by the m controllers:

$$\Sigma_m \quad : \quad \begin{aligned} \begin{bmatrix} \hat{x}_+^{m,1} \\ \vdots \\ \hat{x}_+^{m,m} \end{bmatrix} &= \begin{bmatrix} A(\delta) + B_2(\delta)K_1 & & \\ & \ddots & \\ & & A(\delta) + B_2(\delta)K_m \end{bmatrix} \begin{bmatrix} \hat{x}_k^{m,1} \\ \vdots \\ \hat{x}_k^{m,m} \end{bmatrix} + \begin{bmatrix} B_1/m \\ \vdots \\ B_1/m \end{bmatrix} \hat{w} \\ \hat{z}^m &= [C(\delta) + D_2(\delta)K_1, \ldots, C(\delta) + D_2(\delta)K_m] \begin{bmatrix} \hat{x}_k^{m,1} \\ \vdots \\ \hat{x}_k^{m,m} \end{bmatrix} + D_1(\delta)\hat{w} \end{aligned}$$

where $\hat{x}^{m,i} \in \mathbb{R}^{n_x}$. The index m, i at the right-top corner denotes that the vector is the i-th partition in the state vector of system Σ_m. The following lemma can be easily checked:

Lemma 3. (I/O equivalence) *The original* (1) *and the extended* Σ_m *system are input-output equivalent, i.e.* $[\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_{k-1}] \equiv [w_0, w_1, \ldots, w_{k-1}] \Rightarrow [\hat{z}_1^m, \hat{z}_2^m, \ldots, \hat{z}_k^m] \equiv [z_1, z_2, \ldots, z_k]$ *if* $x_0 = \sum_{i=1}^m \hat{x}_0^{m,i}$ *and* $u_k = \sum_{i=1}^m K_i \hat{x}_k^{m,i}$. *Moreover, if* $u_i(x) = K_i x$ *is stabilizing, so is* $u(x) = \Sigma_{i=1}^m K_i \hat{x}^{m,i}$.

Proof. Applying the control input $u_0 = \sum_{i=1}^m K_i \hat{x}_0^{m,i}$ to (1) and using $x_0 = \sum_{i=1}^m \hat{x}_0^{m,i}$ we get

$$\begin{aligned} x_1 &= A(\delta_k)x_0 + B_2(\delta_k)\sum_{i=1}^m K_i \hat{x}_0^{m,i} + B_1 w_0 = \sum_{i=1}^m \left((A(\delta_k) + B_2(\delta_k)K_i)x_0^{m,i} + \frac{B_1}{m} w_0 \right) = \sum_{i=1}^m \hat{x}_1^{m,i} \\ z_0 &= C(\delta_k)x_0 + D_2(\delta_k)\sum_{i=1}^m K_i \hat{x}_0^{m,i} + D_1(\delta_k)w_0 = \sum_{i=1}^m (C(\delta_k) + D_2(\delta_k)K_i)\hat{x}_0^{m,i} + D_1(\delta_k)w_0 = \hat{z}_0^m \quad (11) \end{aligned}$$

Repeating the computation above for time instants $k > 0$ completes the proof. The stabilizing property of the interpolating controller follows from the stability of subsystems $A(\delta) + B_2(\delta)K_i$. ■

In order to use the interpolating controller $u = \sum_{i=1}^m K_i \hat{x}^{m,i}$ it has to be computable at each time instant k from the actual state and parameter measurements x_k, δ_k. It can be checked this computation can be performed easily in the following way:

$$\begin{aligned} \hat{v}_{k-1}^m \triangleq \frac{1}{m} B_1 w_{k-1} &= \frac{1}{m}[x_k - A(\delta_{k-1})x_{k-1} - B_2(\delta_{k-1})u_{k-1}] \\ \hat{x}_k^{m,i} &= (A(\delta_{k-1}) + B_2(\delta_{k-1})K_i)\hat{x}_{k-1}^{m,i} + \hat{v}_{k-1}^m \\ u_k &= \sum_{i=1}^m K_i \hat{x}_k^{m,i} \end{aligned} \quad (12)$$

Lemma 3. asserts that if system (1) is controlled by (12) then its output is equal to the output of the extended system, i.e. the input-output properties (e.g. the induced $\mathcal{L}_2$ gain) of the closed loop system can be determined from the behavior of the extended system. By exploiting the equivalence, it can be shown in the rest of the section that the partitioning $(\hat{x}_0^{m,1}, \ldots, \hat{x}_0^{m,m})$ of the initial state x_0 can be chosen so that the control policy (12) solves Problem 1.

First, the following lemma is proved:

Lemma 4. ($\mathcal{L}_2$ gain of Σ_m) *The system* Σ_m *has finite* $\mathcal{L}_2$ *gain* $\hat{\gamma}_m \le \sqrt{\sum_{i=1}^m \frac{\gamma_i^2}{m}}$.

Proof. It can easily be seen, there exists a positive definite function $\hat{V}_m : \mathbb{R}^{m \cdot nx} \to \mathbb{R}^+$ and a positive constant $\hat{\gamma}_m$ for system Σ_m s.t. the dissipation inequality

$$(\hat{z}^m)^T \hat{z}^m - \hat{\gamma}_m^2 \hat{w}^T \hat{w} + \hat{V}_m(\hat{x}_+^{m,1}, \ldots, \hat{x}_+^{m,m}) \le \hat{V}(\hat{x}^1, \ldots, \hat{x}^m) \quad (13)$$

is satisfied. For this, consider the subsystems:

$$\begin{aligned} \hat{x}_+^{m,i} &= (A(\delta) + B_2(\delta)K_i)\hat{x}^{m,i} + B_1 \frac{\hat{w}}{m} \\ \hat{z}^{m,i} &= (C(\delta) + D_2(\delta)K_i)\hat{x}^{m,i} + D_1(\delta)\frac{\hat{w}}{m} \end{aligned} \quad (14)$$

which have the property $\hat{z}^m = \sum_{i=1}^m \hat{z}^{m,i}$. Since K_i is a $\mathcal{H}_\infty$ controller, the subsystem above has $\mathcal{L}_2$ gain γ_i, therefore the inequality

$$(\hat{z}^{m,i})^T \hat{z}^{m,i} - \frac{\gamma_i^2}{m^2} \hat{w}^T \hat{w} + \hat{V}_{m,i}(\hat{x}_+^{m,i}) \le \hat{V}_{m,i}(\hat{x}^{m,i}) \quad (15)$$

holds for all $\hat{w} \in W$. Summing up the inequalities above and multiplying the result by m the following relation is obtained

$$m \sum_{i=1}^m (\hat{z}^{m,i})^T \hat{z}^{m,i} - \sum_{i=1}^m \frac{\gamma_i^2}{m} \hat{w}^T \hat{w} + m \sum_{i=1}^m \hat{V}_{m,i}(\hat{x}_+^{m,i}) \le m \sum_{i=1}^m \hat{V}_{m,i}(\hat{x}^{m,i}) \quad (16)$$

Consider now the following inequality defined over arbitrary vectors $v_1, v_2, \ldots, v_m$

$$(v_1 + v_2 + \ldots + v_m)'(v_1 + v_2 + \ldots + v_m) \le m(v_1'v_1 + v_2'v_2 + \ldots + v_m'v_m) \tag{17}$$

(the proof can be found in the Appendix). Substituting $v_i = \hat{z}^{m,i}$ into (17) the following lower bound can be calculated for (16):

$$\begin{aligned}(\hat{z}^m)^T \hat{z}^m - \sum_{i=1}^m \frac{\gamma_i^2}{m}\hat{w}^T\hat{w} + m\sum_{i=1}^m \hat{V}_{m,i}(\hat{x}_+^{m,i}) &\le m\sum_{i=1}^m \left(\hat{z}^{m,i}\right)^T \hat{z}^{m,i} - \sum_{i=1}^m \frac{\gamma_i^2}{m}\hat{w}^T\hat{w} + m\sum_{i=1}^m \hat{V}_{m,i}(\hat{x}_+^{m,i}) \\ &\le m\sum_{i=1}^m \hat{V}_{m,i}(\hat{x}^{m,i})\end{aligned} \tag{18}$$

Note that (18) is the same as (13) with $\hat{\gamma}_m^2 = \sum_{i=1}^m \frac{\gamma_i^2}{m}$ and $\hat{V}_m(\hat{x}^1, \ldots, \hat{x}^m) = m\sum_{i=1}^m \hat{V}_{m,i}(\hat{x}^{m,i})$. ■

Remark. 1. The performance value $\sqrt{\sum_{i=1}^m \frac{\gamma_i^2}{m}}$ is only an upper bound in general for the real $\mathcal{L}_2$ gain. A better approximation can be found by setting $\hat{V}_m = x^T P x$, substituting the system dynamics Σ_m into (13) and rewriting the inequality obtained into an equivalent linear matrix inequality, which is then solved for variables P and $\hat{\gamma}_m$ by convex programming.

It can be easily checked that the number m of the controllers and the controllers themselves in (10) can be chosen so that $\hat{\gamma}_m = \sqrt{\sum_{i=1}^m \frac{\gamma_i^2}{m}} \le \gamma^*$ holds even if $\gamma_m \gg \gamma^*$. This means that a set of 'proper' controllers (having $\mathcal{L}_2$ gain $\le \gamma^*$) can be completed with an 'auxiliary' controller, which does not satisfy the performance specification $\gamma \le \gamma^*$. The next theorem proves it is advisable to choose this 'auxiliary' controller so that it generates 'large' maximal d-invariant set, since it can be achieved – by appropriate partitioning of the initial state – that this set determines the d-invariant set – and consequently the region of applicability – of the interpolating controller (12).

Theorem 1 (d-invariance of interpolation) *Assume $\hat{\gamma}_m = \sqrt{\sum_{i=1}^m \frac{\gamma_i^2}{m}} \le \gamma^*$. Let $\overline{S}_m$ be the maximal robust disturbance invariant set of the closed loop system*

$$x_+ \quad = \quad [A(\delta) + B_2(\delta)K_m]x + B_1 w \tag{19}$$

and let $\underline{S}_i$, $i < m$ be the minimal robust disturbance invariant sets of the closed loop systems

$$x_+ \quad = \quad [A(\delta) + B_2(\delta)K_i]x + \frac{B_1}{m}w \quad i = 1 \ldots m-1 \tag{20}$$

If a scaling factor $0 < \lambda < 1$ is chosen so that the following relations hold

$$x_0 \in \lambda\overline{S}_m \tag{21a}$$

$$\lambda\overline{S}_m \oplus \sum_{i=1}^{m-1} \underline{S}_i \subset \overline{S}_m \tag{21b}$$

$$\frac{1}{\lambda m} < 1 \tag{21c}$$

then in case of initial partitioning

$$\hat{x}_0^{m,1} = \hat{x}_0^{m,2} = \ldots = 0 \qquad \hat{x}_0^{m,m} = x_0 \tag{22}$$

the set $\overline{S}_m$ is a d-invariant set of the closed loop system controlled for all $w \in W$. ($\oplus$ denotes the Minkowski sum of sets.)

Proof. By relation (21/b) it is enough to prove that conditions (21/a,c) imply $\hat{x}^{m,i} \in \underline{S}_i$ and $\hat{x}^{m,m} \in \lambda\overline{S}_m$ for all time instants. Since $\hat{x}_0^{m,i} = 0 \in \underline{S}_i$ and $\hat{x}_0^{m,m} = x_0 \in \lambda\overline{S}_m$ it is enough to show $\hat{x}_+^{m,i} \in \underline{S}_i$ and $\hat{x}_+^{m,m} \in \lambda\overline{S}_m$ follows from $\hat{x}^{m,i} \in \underline{S}_i$ and $\hat{x}^{m,m} \in \lambda\overline{S}_m$. Consider first $\hat{x}_+^{m,i}$:

$$\hat{x}_+^{m,i} = [A(\delta) + B_2(\delta)K_i]\hat{x}^{m,i} + \frac{B_1}{m}w \tag{23}$$

By the d-invariance of $\underline{S}_i$, $\hat{x}^{m,i} \in \underline{S}_i$ immediately implies $\hat{x}_+^{m,i} \in \underline{S}_i$. In the case of $\hat{x}^{m,m}$ the reasoning is similar. We start from the dynamic equation

$$\hat{x}_+^{m,m} = [A(\delta) + B_2(\delta)K_m]\hat{x}^{m,m} + \frac{B_1}{m}w \tag{24}$$

Since $(1/\lambda) \cdot \hat{x}^{m,m} \in \overline{S}_m$ thus, by the definition of the invariant set,

$$[A(\delta) + B_2(\delta)K_m]\frac{\hat{x}^{m,m}}{\lambda} + B_1 w \in \overline{S}_m \tag{25}$$

Dividing equation (24) by λ we get that

$$\frac{\hat{x}_+^{m,m}}{\lambda} = [A(\delta) + B_2(\delta)K_m]\frac{\hat{x}^{m,m}}{\lambda} + \frac{B_1}{\lambda m}w \tag{26}$$

Considering that W contains the origin in its interior thus $\frac{\hat{x}_+^{m,m}}{\lambda} \in \overline{S}_m$ follows from (25) only if $\frac{1}{\lambda m} < 1$, which is equal to condition (21/c) so it holds by assumption. ■

Remark. 2. By condition (21/a) the initial state has to be inside $\lambda\overline{S}_i$ in order that the constraint satisfaction can be guaranteed. Note at the same time that the sets $\underline{S}_i$, which have to be subtracted from $\overline{S}_m$, are *minimal* d-invariance sets belonging to a closed loop system (23) seeing only the m-th of the disturbance. Therefore these sets are quite 'small' compared to the *maximal* d-invariant set $\overline{S}_m$. Consequently the invariant set $\lambda\overline{S}_m$ of the interpolating controller remains much larger than the maximal d-invariance set of any other individual K_i controller, $i < m$.

Remark. 3. It is possible to use the interpolation based control with only two controllers. In this case one controller has to satisfy the performance condition (its induced $\mathcal{L}_2$ gain $\leq \gamma^*$) while the second has to generate large d-invariant set. In the construction of the interpolating control and in the extended system the 'good' controller has to be repeated $m-1$ times where m is chosen so that $\hat{\gamma}_m = \sqrt{\sum_{i=1}^m \frac{\gamma_i^2}{m}} \leq \gamma^*$ holds.

Remark. 4. The conditions of Theorem 1 can be slightly relaxed by prescribing only the existence of partitioning $\sum_{i=1}^m \hat{x}_0^{m,i} = x_0$ s.t.

$$\hat{x}_0^{m,1} \in \underline{S}_1, \ldots, \hat{x}_0^{m,m-1} \in \underline{S}_{m-1}, \hat{x}_0^{m,m} \in \lambda\overline{S}_m \tag{27}$$

In this case not the state x_0, but its m-th partition $\hat{x}_0^{m,m}$ is required only to be contained by the set $\lambda\overline{S}_m$. If the minimal d-invariant sets $\underline{S}_i$ are small (as it is the case in the numerical example in section 6) (27) does not provide significant advantage.

5 On-line performance improvement

It has been proved, the interpolation based controller (12) is able to solve Problem (1) and consequently it is valid over a larger invariant set, than the separately applied $\mathcal{H}_\infty$ controllers based on it was built up. Since the interpolation scheme, with the the control structure, is fixed at time 0 (by partitioning), the performance of the control does not change later. Thus, the induced $\mathcal{L}_2$ gain remains $\hat{\gamma}_m$ on the entire infinite horizon.

In the section it is shown that the performance of the control can be improved (the induced $\mathcal{L}_2$ gain can be decreased) by modifying the control structure on-line.

Construct first, m extended systems $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ from the closed loop dynamics formed by the first m controllers. These systems define m different interpolating controllers $u_n(x) = \sum_{i=1}^n K_i x$, $n \in \{1, 2, \ldots, m\}$. By Lemma 4. $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ have finite $\mathcal{L}_2$ gains and by construction they preserve the original ordering: $\hat{\gamma}_1 \leq \hat{\gamma}_2 \leq \ldots \leq \hat{\gamma}_m$.

The control starts then as before: the initial state is partitioned according to (22) and the system is controlled by $u_m(x) = \sum_{i=1}^m K_i x^{m,i}$. The induced $\mathcal{L}_2$ gain of the closed loop is of course $\hat{\gamma}_m$. When the trajectory enters later into a d-invariant set of the next – the $(m-1)$-th – controller $u_{m-1}(x)$ then the state is repartitioned among the first $m-1$ controllers, i.e. the m-th controller is dropped. Since from this time instant the closed loop system is I/O equivalent to the extended system Σ_{m-1} and $\hat{\gamma}_{m-1} < \hat{\gamma}_m$ thus the performance is improved. When the trajectory enters into the next d-invariant set this procedure can be repeated. In the end, only one (the best) controller remains. The next theorem tells how this repartitioning has to be correctly performed.

Theorem 2 (performance update) *Let $D_2 \equiv 0$. Assume $n \leq m$ controllers[4] are active currently, (i.e. the last control input was computed by $u_{k-1} = \sum_{i=1}^{n} K_i \hat{x}_{k-1}^{n,i}$). If at the current time k there exists $0 < \lambda < 1$ s.t.*

$$x_k \in \lambda \overline{S}_{n-1}, \quad \lambda \overline{S}_{n-1} \oplus \textstyle\sum_{i=1}^{n-2} \underline{S}_i \subseteq \overline{S}_{n-1}, \quad \frac{1}{\lambda(n-1)} < 1 \tag{28}$$

where $\overline{S}_{n-1}$ and $\underline{S}_i$ are the maximal and minimal d-invariant sets for systems $[A(\rho)+B_2(\rho)K_n]x + B_1 w$ and $[A(\rho)+B_2(\rho)K_i]x + \frac{B_1}{m} w$, respectively. Moreover if

$$\hat{V}_{n-1}(0, \dots, 0, x_k) \leq \hat{V}_n(\hat{x}_k^{n,1}, \dots, \hat{x}_k^{n,n}) \tag{29}$$

then after repartitioning the state as $\hat{x}_k^{n-1,1} = 0, \dots, \hat{x}_k^{n-1,n-2} = 0, \hat{x}_k^{n-1,n-1} = x_k$ the control input $u_\ell = \sum_{i=1}^{n-1} K_i \hat{x}_\ell^{n-1,i}$, $\ell \geq k$ produces a closed loop system, which has induced $\mathcal{L}_2$ gain $\hat{\gamma}_{n-1}$ and has trajectories running inside the set $\overline{S}_{n-1}$.

Proof. The constraint satisfaction follows from the conditions (28) in the same way as in Theorem 1. We prove only that the $\mathcal{L}_2$ gain is at most $\hat{\gamma}_m$. Before time k the dissipation inequalities hold with $\hat{V}_n$ and $\hat{\gamma}_n$:

$$\begin{aligned}
(\hat{z}_0^m)^T \hat{z}_0^m - \hat{\gamma}_m^2 \hat{w}_0^T \hat{w}_0 + \hat{V}_m(\hat{x}_1^m) &\leq \hat{V}_m(\hat{x}_0^m) \\
&\vdots \\
(\hat{z}_{k-2}^n)^T \hat{z}_{k-2}^n - \hat{\gamma}_n^2 \hat{w}_{k-2}^T \hat{w}_{k-2} + \hat{V}_n(\hat{x}_{k-1}^n) &\leq \hat{V}_n(\hat{x}_{k-2}^n) \\
(\hat{z}_{k-1}^n)^T \hat{z}_{k-1}^n - \hat{\gamma}_n^2 \hat{w}_{k-1}^T \hat{w}_{k-1} + \hat{V}_n(\hat{x}_k^n) &\leq \hat{V}_n(\hat{x}_{k-1}^n)
\end{aligned} \tag{30}$$

at time k:

$$(\hat{z}_k^n)^T \hat{z}_k^n - \hat{\gamma}_n^2 \hat{w}_k^T \hat{w}_k + \hat{V}_{n-1}(\hat{x}_{k+1}^{n-1}) \leq (\hat{z}_k^{n-1})^T \hat{z}_k^{n-1} - \hat{\gamma}_{n-1}^2 \hat{w}_k^T \hat{w}_k + \hat{V}_{n-1}(\hat{x}_{k+1}^{n-1}) \leq \hat{V}_{n-1}(\hat{x}_k^{n-1}) \leq \hat{V}_n(\hat{x}_k^n) \tag{31}$$

and later:

$$\begin{aligned}
(\hat{z}_{k+1}^{n-1})^T \hat{z}_{k+1}^{n-1} - \hat{\gamma}_n^2 \hat{w}_{k+1}^T \hat{w}_{k+1} + \hat{V}_{n-1}(\hat{x}_{k+2}^{n-1}) \leq (\hat{z}_{k+1}^{n-1})^T \hat{z}_{k+1}^{n-1} - \hat{\gamma}_{n-1}^2 \hat{w}_{k+1}^T \hat{w}_{k+1} + \hat{V}_{n-1}(\hat{x}_{k+2}^{n-1}) &\leq \hat{V}_{n-1}(\hat{x}_{k+1}^{n-1}) \\
&\vdots
\end{aligned} \tag{32}$$

The leftmost inequality in (31) holdss because $\hat{\gamma}_{n-1} \leq \hat{\gamma}_n$ and $\hat{z}_k^{n-1} = \hat{z}_k^n$ (since $D_2 = 0$), the middle one holds, since Σ_{n-1} has $\mathcal{L}_2$ gain $\hat{\gamma}_{n-1}$ and the rightmost inequality holds due to (29). Summing up the inequalities (30),(31),(32) from 0 to a sufficiently large N we get that

$$\sum_{i=0}^{N} z^T z - \hat{\gamma}_m^2 \sum_{i=0}^{N} w^T w \leq \hat{V}_m(\hat{x}_0^m) - \hat{V}_1(\hat{x}_{N+1}^1) \tag{33}$$

Since (12) is stabilizing $\hat{V}_1(\hat{x}_{N+1}^1) \to 0$ as $N \to \infty$. Therefore the $\mathcal{L}_2$ gain of the closed loop system is not worse than $\hat{\gamma}_m$, the performance of the first control policy. ∎

6 Numerical example

The system to be controlled is defined by the following system matrices:

$$\begin{aligned}
&A_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.3 & 1.2 \end{bmatrix} \; A_2 = \begin{bmatrix} 1.1 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \; B_1 = \begin{bmatrix} 0.15 \\ 1 \end{bmatrix} \; B_{2_1} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} \; B_{2_2} = \begin{bmatrix} 0.2 \\ 1.5 \end{bmatrix} \\
&C_1 = \begin{bmatrix} 0.6 & 0.3 \end{bmatrix} \; C_2 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \; D_{1_1} = -0.3, \;\; D_{1_2} = 0.3, \;\; D_2 = 0
\end{aligned} \tag{34}$$

[4] Of course the system starts with $n = m$ at time 0

Let the polytopes X, U, W be given as follows

$$X = \{x \mid \begin{bmatrix} +1 & 0 \\ -1 & 0 \\ 0 & +1 \\ 0 & -1 \end{bmatrix} x \leq 5\}, \quad U = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} -0.3 & 0.3 \end{bmatrix} \tag{35}$$

It is prescribed moreover that the induced $\mathcal{L}_2$ gain between w and z has to be smaller than $\gamma^* = 1$.

Two alternate $\mathcal{H}_\infty$ controllers K_a and K_b have been designed independently with the following performance list:

$$\gamma_{a,b} = \begin{bmatrix} 0.6093 & 2.0646 \end{bmatrix} \tag{36}$$

The first controller was obtained by solving the convex optimization in A.1. The second one is a simple quadratically stabilizing controller providing finite $\mathcal{L}_2$ gain. Note, the second controller has an unacceptable performance it can not be applied in itself. On the other hand, the closed loop d-invariant set associated to K_b is much larger than the set that corresponds to K_a.

An extended system was constructed by repeating the K_a controller 7 times, i.e. $K_1 = K_2 = \ldots K_7 = K_a$ and using K_b as K_8. The $\mathcal{L}_2$ gain computed by Lemma 4. is $\sqrt{\sum_{i=1}^{8} \frac{\gamma_i^2}{8}} = 0.9261$. The gain according to Remark 1. has been reevaluated and obtained $\hat{\gamma}_8 = 0.7721 (< \gamma^* = 1)$.

The invariant sets of interest are depicted in Figure 2. The largest set $\overline{S}_b (= \overline{S}_8)$ is the maximal d-invariant set generated by controller $K_b (= K_8)$. The smallest invariant set is the $\underline{S}_a (= \underline{S}_i,\ i \leq 7)$ related to the controller $K_a (= K_i,\ i \leq 7)$. $\overline{S}_a (= \overline{S}_i,\ i \leq 7)$ is contained in $\overline{S}_b$ as well. The d-invariant set (the region of applicability) of the interpolation based controller was drawn by the bold invariant set $(\lambda \overline{S}_b)$. The set was determined by (21), satisfying $\lambda \overline{S}_b \subset= \overline{S}_b \ominus 7 \underline{S}_a$. In our case $\lambda = 0.65$ has been chosen.

Furthermore, a K_c controller has been designed to assure approximately the same performance as it is given by the interpolated controllers. Thus, K_c gives a maximal domain of applicability for a single constrained $\mathcal{H}_\infty$ controller. The set $\overline{S}_c$ in Figure 2 is the maximal d-invariant set of a linear state feedback controller $u(x) = K_c x$ providing approximately the same performance level $(\gamma_c = 0.7763)$ as the interpolating controller has. It can be seen that the set $\lambda \overline{S}_b$ is *larger* than the set $\overline{S}_c$.

The region of applicability of the constrained $\mathcal{H}_\infty$ controller (defined by a single state feedback law) could be enlarged using the interpolation based control method. On the other hand, the shape of the d-invariance set generated by the interpolating controller inherits the shape of the d-invariant set belonging to the'auxiliary' controller (K_b), therefore the explicit relation $\overline{S}_c \subset \lambda \overline{S}_m$ can not always be guaranteed.

After constructing the invariant sets the simulation was started at $x_0 = [-2.5\ ;\ 2]$. The parameter δ^1 was changing according to Figure 4 $(\delta^2 = 1 - \delta^1)$. The control input, system output and the trajectory can be seen in the figures 3,5. The on-line performance update was now switched off, all of the 8 controllers were active during the simulation.

The results of the simulation performed with performance update are depicted in figures 7, 8. At the 5-th time step the trajectory entered into the maximal d-invariant set of controller K_a. By examining figure 6, which shows the values of function $\hat{V}_1$ and $\hat{V}_8$ respectively, one can check that $\hat{V}_1$ was smaller at this step than $\hat{V}_8$, thus by Theorem 2 the interpolation based controller could be replaced by the single state feedback K_a. From this step the performance of the control was equal to $\gamma_a (= 0.6093)$, the performance of K_a. In this example this performance improvement was not prominently significant, since the interpolation based controller provided in itself a good performance $(\hat{\gamma}_8 = 0.7721)$.

7 Conclusion

The conservatism of a single linear constrained $\mathcal{H}_\infty$ controller for LPV systems has been studied in the paper. A novel methodology proposes the interpolation based controller strategy to enlarge the robust d-invariant set of the closed loop system for polytopic parameter dependent plants. The interpolation not only extends the region of applicability, but also provides online performance update possibilities.

In the paper, an interpolation based, constrained $\mathcal{H}_\infty$ control method has been proposed for discrete-time, LPV systems. It is proved by interpolating among the appropriately chosen linear feedback controllers, an extended control policy is achieved over a larger domain than a single linear feedback. Furthermore, the

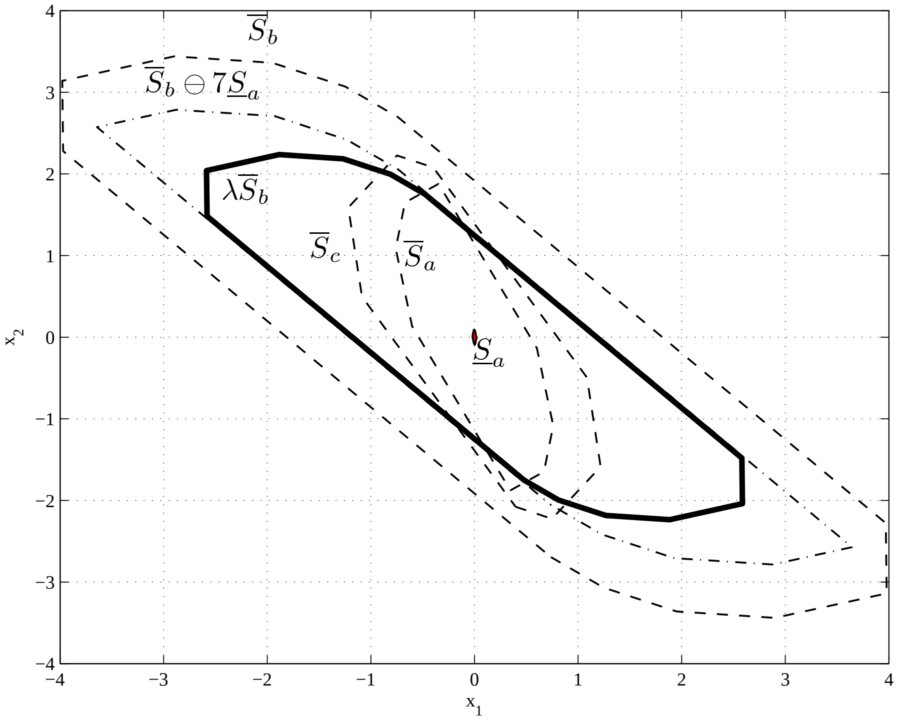

Fig. 2: The invariant sets generated by using K_a, K_b and K_c

region of applicability of the proposed method can be calculated easily from the d-invariant sets generated by the feedback controllers used in the interpolation. The d-invariant sets are computed by linear programming directly from the system dynamics and the constraints. This provides a less conservative results than other methods using approximations based on the level sets of the storage function. The applicability of the method was tested and demonstrated on a simple, numerical example.

In contrast to other MPC based approaches, the presented method does not require on-line optimization, therefore it can be used in real-time.

Our method is based on parameter independent controllers and all results were derived with assuming constant B_1 matrix. In some cases these restrictions can be relaxed. The presented algorithms can be easily extended to the case of parameter varying B_1 and if B_2 is constant, one may even use parameter varying $K_i(\rho)$ controllers. Nevertheless, the general case still has not been covered, when all system matrices are parameter dependent, the system to be controlled is uncertain and/or the controllers are also parameter varying or possibly dynamic. To extend the results to these more general situations may be one of the main direction of the future research.

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A Appendix

A.1 LMI formulation of the unconstrained $\mathcal{H}_\infty$ problem

The unconstrained $\mathcal{H}_\infty$ control problem (part) of Problem (1) can be solved easily by convex optimization in case of $u(x) = Kx$ and $V(x) = x^T Px$. For this, assume there exists $Q > 0, Y$ and $\gamma > 0$ solving the

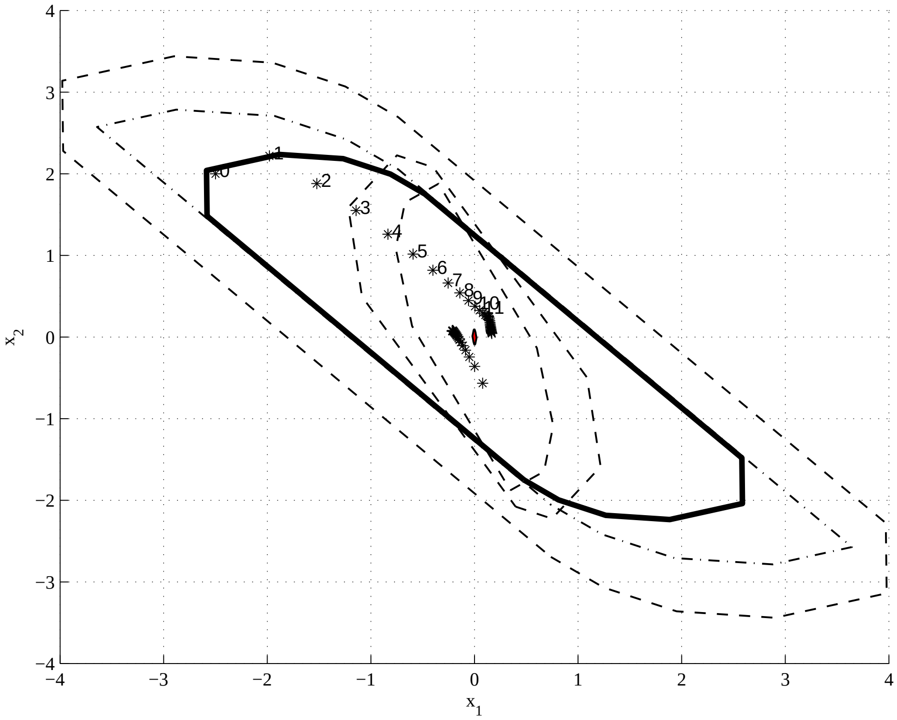

Fig. 3: Trajectory of the controlled system in case of no performance update.

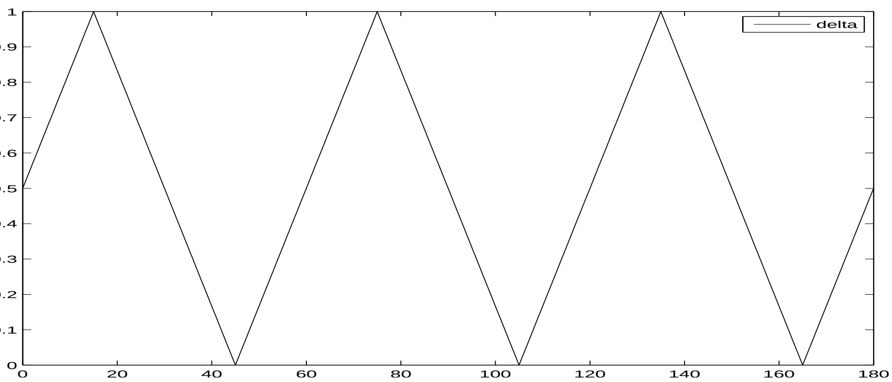

Fig. 4: The scheduling parameter δ^1. ($\delta^2 = 1 - \delta^1$)

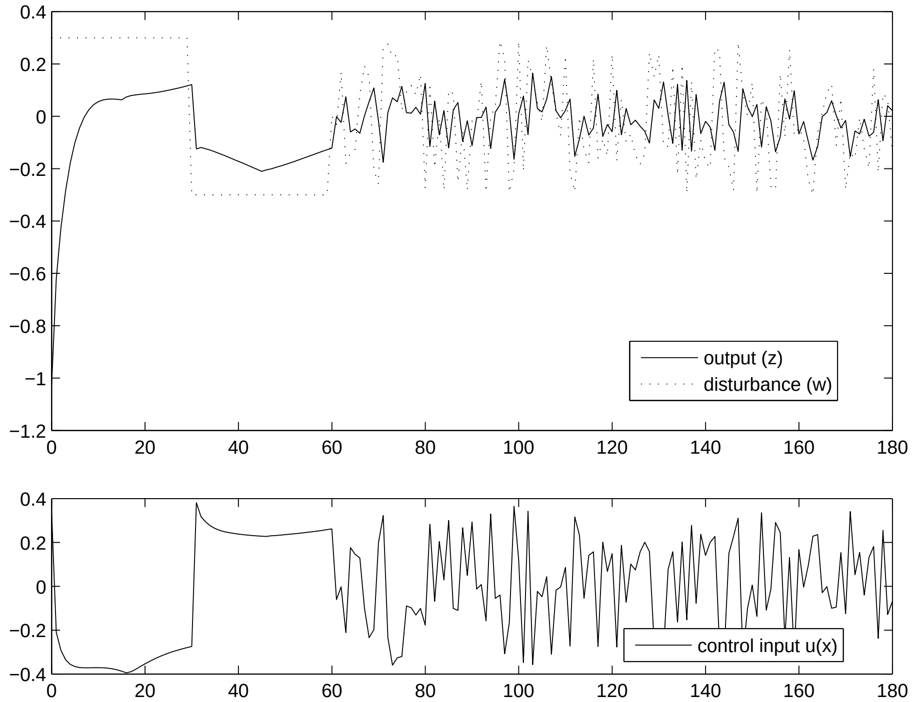

Fig. 5: The system output z, the disturbance w (dotted) and the control input applied in case of no performance update.

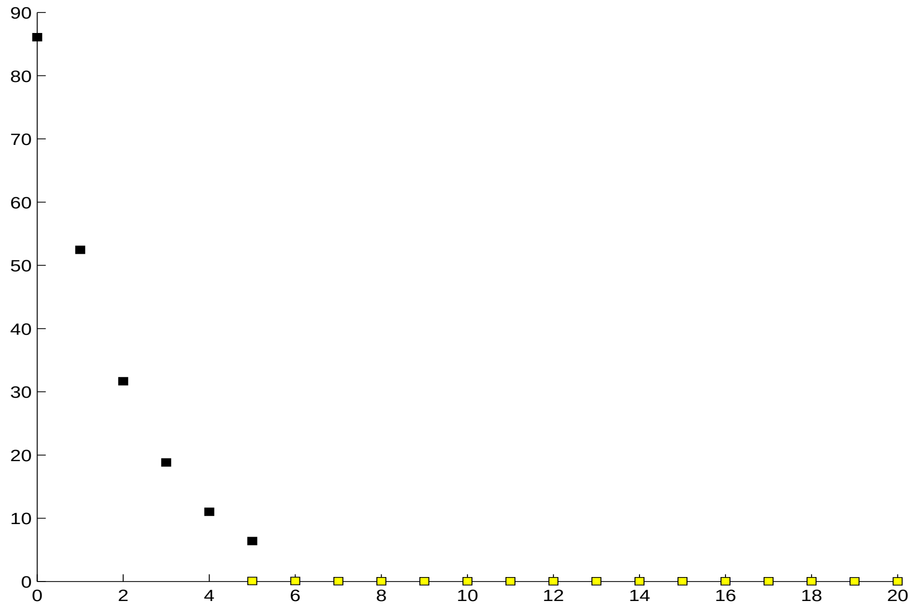

Fig. 6: The values of $\hat{V}_8$ (in the first 5 steps, dark squares) and the values of $\hat{V}_1$ (from the 5th step, light squares).

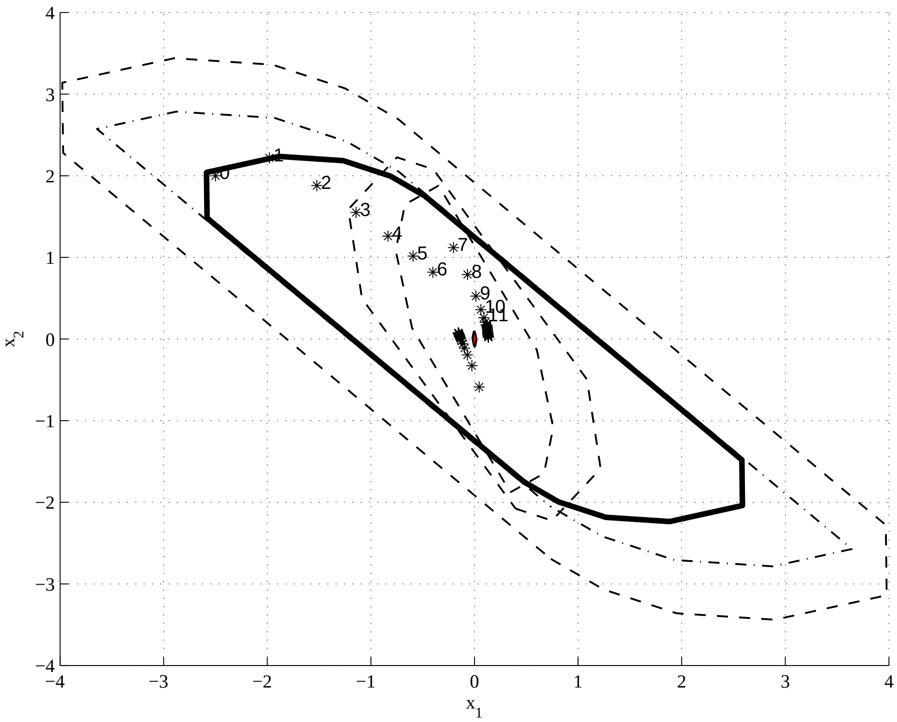

Fig. 7: Trajectory of the controlled system in case of performance update.

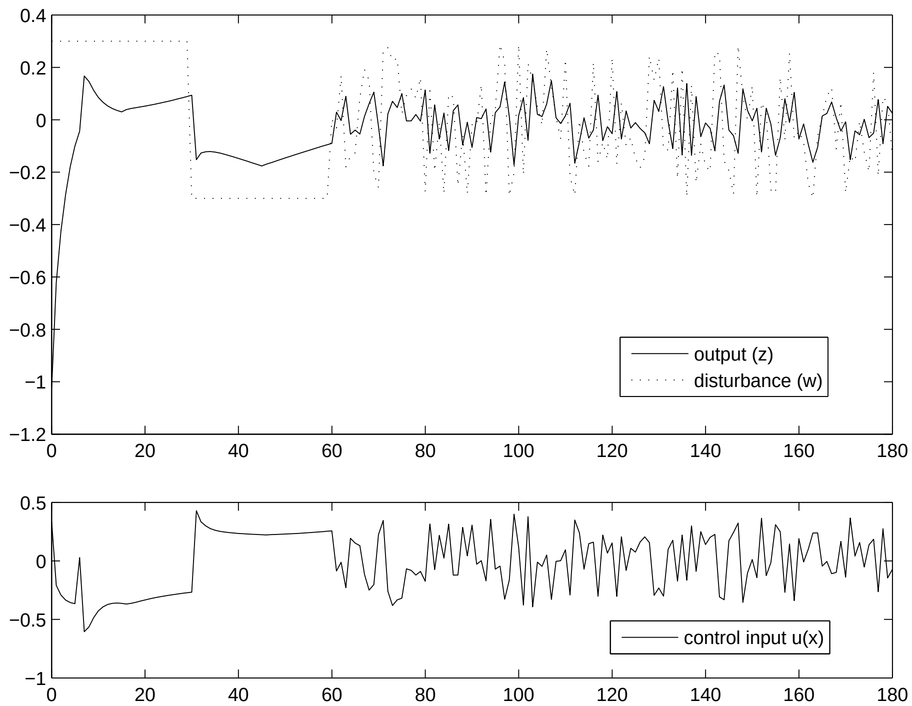

Fig. 8: The system output z, the disturbance w (dotted) and the control input applied in case of performance update.

following linear objective minimization problem:

$$\begin{gathered}\min_{Q,Y}\gamma\\ \gamma>0\\ \begin{bmatrix} Q & & * & * \\ & \gamma^2 I & * & * \\ A_iQ+B_{2i}Y & B_{1i} & Q & \\ C_iQ+D_{2i}Y & D_{1i} & & I \end{bmatrix}>0\\ i=1...L\end{gathered} \tag{37}$$

then $P=Q^{-1}$, $K=Q^{-1}Y$ solve the unconstrained $\mathcal{H}_\infty$ control problem with performance γ. (For the details see e.g. [10].)

A.2 Proof of inequality (17)

Let $v_1, v_2, \ldots, v_m$ denote m arbitrary vectors. The following inequalities can be easily checked:

$$\begin{aligned} 0 &\leq \sum_{i<j}(v_i-v_j)^T(v_i-v_j)\\ 2\sum_{i<j}v_i^Tv_j &\leq (m-1)\sum_i v_i^Tv_i\\ 2\sum_{i<j}v_i^Tv_j+\sum_i v_i^Tv_i &\leq m\sum_i v_i^Tv_i \end{aligned} \tag{38}$$

Since $2\sum_{i<j}v_i^Tv_j+\sum_i v_i^Tv_i=(v_1+\ldots+v_m)^T(v_1+\ldots+v_m)$ and $m\sum_i v_i^Tv_i=m(v_1^Tv_1+\ldots+v_m^Tv_m)$ thus the inequality (17) follows.

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