ANALYSIS AND CONTROL OF QUASI-POLYNOMIAL DAE SYSTEMS

Theses of Ph.D. dissertation

written by

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1 Motivation and aim

The analysis and control of nonlinear dynamical systems is an undoubtedly challenging field of systems science. A large number of concentrated parameter nonlinear systems can be mathematically described by finite dimensional nonlinear *input-affine* state space models, where the state equation is a set of ordinary differential equations (ODEs) [1]. This is a definitely advantageous kind of representation with well founded tools for dynamical (stability, controllability, observability) analysis and also for controller design [2],[3],[4].

However, concentrated parameter dynamical systems are usually modelled by a *mixed* set of differential *and* algebraic equations (DAEs) using a systematic modelling procedure [1]. To achieve the advantageous purely differential form, the algebraic equations are then substituted to the differential ones. It occurs quite often that this elimination cannot be performed and the state equation remains in DAE form. Unfortunately, only a few analysis and controller design tools can be found in the literature, moreover they generally deal with only very narrow classes of DAEs [5], [6].

Since a rising number of dynamical models - e. g. models of strongly nonlinear complex systems or high precision models that have an emerging role in a lot of practical fields with an ever-growing demand for punctuality - fall into the class of non-substitutable DAE models, the lack of widely applicable general analysis and controller design methods is a serious and emerging problem that calls for a solution.

This motivates us to alloy the DAE form with another formalism, namely the quasi-polynomial (QP) form, that has been proved useful in the analysis and control of systems with ODE models [7],[8]. Under the mild condition of smoothness, the QP-DAE form can be regarded as a general mathematical representation of lumped parameter systems.

This thesis represents the very first steps towards the analysis and control of QP-DAE systems. Three related topics will be presented that are common in attempting to exploit the advantages of the QP formalism.

It is well known from the literature, that there are no generally applicable constructive ways to find an appropriate Lyapunov function to prove the asymptotic stability of a steady state point and to determine the stability neighborhood in the general nonlinear case [9]. However, for a class of rank-deficient QP models, namely Lotka-Volterra models - which are *hidden QP-DAEs* - there is an algorithmic procedure for finding a quadratic Lyapunov function and estimating the stability neighborhood of a steady state

point [10]. The first topic aims to *apply* this method to different zero dynamics of a strongly nonlinear QP model of a low-power gas turbine.

The second topic is directed towards designing controllers of different type for a low power gas turbine. This topic is QP-specific in the sense that the gas turbine model is a QP-DAE with substitutable algebraic equations. Although standard controller design methods are to be applied, the special QP structure is to be exploited again.

Finding constants of motion (invariants) of ODE systems is a complex topic of mathematics. Furthermore, it has a great theoretical and practical importance in systems and control theory [3]. The last topic aims to develop a new, algorithmizable method to find QP type invariants of QP state space models.

2 Methods and tools

In this section, the most important notions and tools used in the Thesis are briefly summarized.

2.1 System representations

State equations in the form of ordinary differential equations (ODEs)
Denote the vector of states by $x \in \chi \subset \mathbb{R}^n$, the vector of system inputs and outputs by $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$, respectively. The general nonlinear input-affine form consists of an ODE state and an output equation [11]:

$$\dot{x} = \frac{dx}{dt} = f(x) + \sum_{i=1}^{p} g_i(x)u_i \tag{1}$$

$$y = h(x) \tag{2}$$

where $f, g_i \in \mathbb{R}^n \mapsto \mathbb{R}^n$, $i = 1, \ldots, p$ and $h \in \mathbb{R}^n \mapsto \mathbb{R}^q$ are smooth nonlinear functions, and $u = [u_1, \ldots, u_p]^T$.

A function $I : \mathbb{R}^n \mapsto \mathbb{R}$ is called an **invariant** (constant of motion, first integral or hidden algebraic constraint) of the ODE in (1) if

$$\frac{d}{dt}I = \frac{\partial I}{\partial x} \cdot \dot{x} = 0.$$

State equations in the form of differential algebraic equations (DAEs)
The following, input-affine semi-explicit DAE [12] can be regarded as *the*

general representation form of lumped parameter dynamical systems:

$$\dot{x} = f(x,z) + \sum_{i=1}^{p} g_i(x,z)u_i$$
$$0 = w(x,z)$$
$$y = h(x,z)$$

where $x \in \mathbb{R}^{n_1}$ and $z \in \mathbb{R}^{n_2}$ are the vectors of the differential and the algebraic state variables, respectively, $f, g_i \in \mathbb{R}^{n_1+n_2} \mapsto \mathbb{R}^{n_1}$, $i = 1, \ldots, p$, $w \in \mathbb{R}^{n_1+n_2} \mapsto \mathbb{R}^{n_2}$ and $h \in \mathbb{R}^{n_1+n_2} \mapsto \mathbb{R}^{q}$ are smooth nonlinear functions, and $u = [u_1, \ldots, u_p]^T$ is the vector of control input variables.

Quasi-polynomial (QP) models

The majority of nonlinear autonomous ODE models (i.e. (1)-(2) with $u = 0$ or with $u = \phi(x)$) with smooth nonlinearities can be algorithmically transformed to the so-called quasi-polynomial (QP) form [13]:

$$\dot{x}_i = x_i\left(\lambda_i + \sum_{j=1}^{m} A_{ij}U_j\right), \quad x_i > 0, \quad i = 1, \ldots, n, \quad m \geq n \quad (3)$$
$$y = h(x) \quad (4)$$

where the terms U_j are the *quasi-monomials* (QMs) of the system:

$$U_j = \prod_{k=1}^{n} x_k^{B_{jk}}, \quad j = 1, \ldots, m$$

and $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times n}$, $\lambda \in \mathbb{R}^n$ are constant matrices and vectors.

The **Lotka-Volterra (LV) form of the QP state equation (3)** is [13]:

$$\dot{U}_\ell = U_\ell\left(\lambda_{LV_\ell} + \sum_{j=1}^{m} A_{LV_{\ell,j}}U_j\right), \qquad \ell = 1, \ldots, m \quad (5)$$

where the elements of the state vector are exactly the QMs: $U = [U_1, \ldots, U_m]^T$, $\lambda_{LV} = B \cdot \lambda \in \mathbb{R}^m$ and $A_{LV} = B \cdot A \in \mathbb{R}^{m \times m}$. LV models computed from QP models are generally non-minimal (when $m > n$), therefore their system trajectories can evolve on an n dimensional manifold (surface) of $\mathbb{R}^m$.

2.2 Stability analysis: local quadratic stability of LV systems

Denote U^* a steady state of the state variable U of an LV state equation (5), and let $\overline{U} = U - U^*$. The general quadratic Lyapunov function candidate for

this system and its time-derivative are:

$$V(\overline{U}) = \overline{U}^T P \overline{U}, \quad P \in \mathbb{R}^{m \times m}, \ P > 0$$

$$\frac{dV}{dt} = \overline{U}^T \left\{ P\langle \overline{U} \rangle A_{LV} + P\langle U^* \rangle A_{LV} + A_{LV}^T \langle \overline{U} \rangle P + A_{LV}^T \langle U^* \rangle P \right\} \overline{U}$$

where $[\overline{U}_1, \ldots, \overline{U}_m]^T \in \mathscr{N} \subset \mathbb{R}^m$, while $\langle \overline{U} \rangle$ and $\langle U^* \rangle$ are diagonal quadratic matrices built from the vectors $\overline{U}$ and U^*, respectively.

The non-increasing nature of the quadratic Lyapunov function in a neighborhood $\mathscr{N}$ of the origin is equivalent to the validity of the following matrix inequality, which is *bilinear* in its variables P and $\langle \overline{U} \rangle$ [10]:

$$P\langle \overline{U} \rangle A_{LV} + P\langle U^* \rangle A_{LV} + A_{LV}^T \langle \overline{U} \rangle P + A_{LV}^T \langle U^* \rangle P \leq 0 \tag{6}$$

Therefore the quadratic stability region can be estimated by first setting $\langle \overline{U} \rangle = 0$ and solving (6) as a *linear matrix inequality (LMI)* [14] for the matrix variable P, and then - with this P - solving (6) as an LMI for the variable $\langle \overline{U} \rangle$. The solutions $\langle \overline{U} \rangle$ define a *convex* neighborhood of the origin [14].

Several different **control techniques** have been used in the Thesis. All the controllers are based on input-output linearization [3] or adaptive feedback linearization [4], and linear quadratic controllers [11] and/or constrained linear optimal controllers [15] are put to the linearized plants.

The identified model of a DEUTZ T216 type low-power **gas turbine** has been used in this Thesis. Its third order nonlinear input-affine model has been developed in [16].

3 New scientific results

The new scientific results presented are summarized in the following Theses.

Thesis 1 *Stability analysis of the zero dynamics of a low power gas turbine model in QP form* (Chapter 3)
(**[P1],[P2],[P3],[P4]**)
The local stability of two different zero dynamics of a third order nonlinear low power gas turbine model taken from literature [16] has been investigated.

For the *zero dynamics for the turbine inlet total pressure* as output, a quadratic stability analysis method for LV systems known from literature [10] has been applied to a typical operating point, with stability neighborhood estimation. The estimated stability neighborhood covers approximately 56 % of the operating domain. Simulations confirmed theoretical results, moreover the trajectories started out of the quadratic stability neighborhood showed the conservativeness of this estimation since the range of stability is proved to be wider than the estimated one. By the application of this stability analysis method to different zero dynamics belonging to different steady states of the turbine inlet total pressure, the results have been generalized and the possible causes of the conservativeness of the estimation method have been discussed.
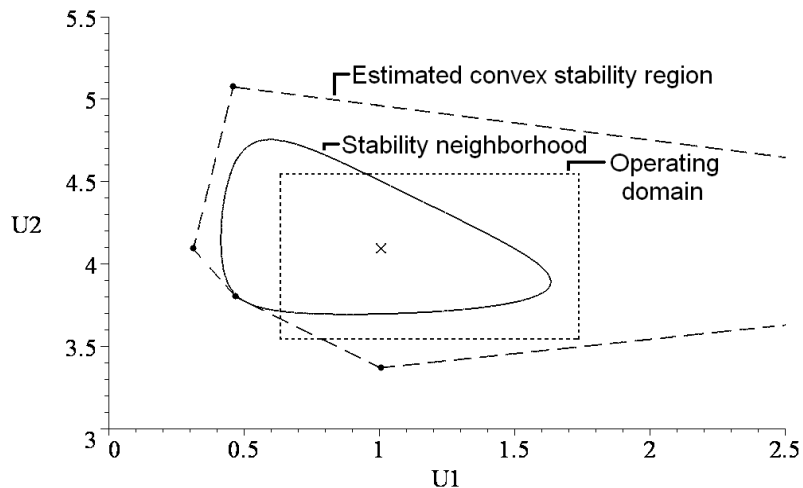

Figure 1: Quadratic stability neighborhood estimation

The stability of the one dimensional *zero dynamics for the rotational speed* has been investigated with phase diagrams. The equilibrium of this zero dynamics has been found to be unique and stable in the whole operating domain, independently of the arbitrary constant values of the

rotational speed and of the load torque, which is the most versatile environmental disturbance.

The results of this Thesis give a basis for control structure selection for the low power gas turbine.

Thesis 2 *Controller design for a low-power gas turbine* (Chapter 4) (**[P2],[P3],[P4],[P5],[P6]**)
The rotational speed of the low power gas turbine has been chosen to be controlled via three different input-output linearization based linear controllers:

(a) An LQ servo controller that tracks a piecewise constant reference signal for the rotational speed, with known time-function of the load torque;

(b) an LQ+MPT controller that keeps the rotational speed at a constant value and the states between bounded values, with measurable load torque;

(c) a novel adaptive LQ servo controller that is an extension of (a): the time evolution of the *load torque is unknown and is estimated*.

Simulation experiments showed that all controllers guarantee robustness against model parameter uncertainties and environmental disturbances.

In cases (a) and (b) the time function of the load torque has been assumed to be known, although in most cases it is an unmeasurable disturbance of the environment. The advanced controller (c) handles this more realistic case with a novel approach: the load torque is estimated by a dynamic feedback supplied by a state space based adaptive controller and used in the input-output linearizing feedback.

Worst case simulations in MATLAB/SIMULINK have showed that the controller properly estimates the time function of the load torque, moreover the reference tracking for the rotational speed is robust against all environmental disturbances and model parameter uncertainties. In spite that the load torque is only estimated, the controlled plant shows the same robustness as (a) with known load torque.

The importance of this result is well characterized by the fact that this approach has not been applied to gas turbines yet, however the load torque is the most important environmental disturbance because of its non-measurability, versatility and impact on the time behavior of the gas turbine.
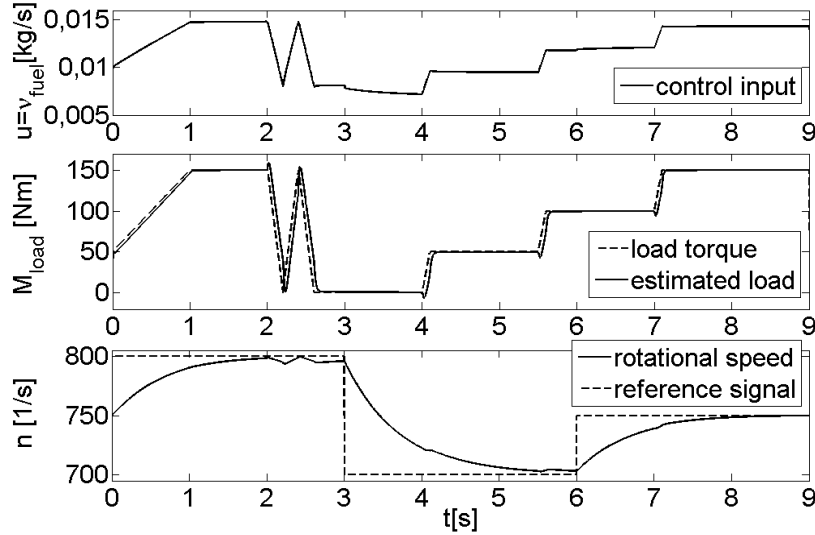
Figure 2: Adaptive estimation of the load torque

Thesis 3 *Determining invariants (first integrals) of QP-ODEs* (Chapter 5)
([P7],[P8],[P9],[P10],[P11])

A new algorithm has been developed that is able to determine a wide class of QP-type invariants of non-minimal QP-ODE systems. The operation of this algorithm is based on simple matrix operations, and does not contain any heuristic steps.

Two versions of this algorithm has been presented: the former is designed to determine single, the latter is to determine multiple first integrals. Both versions of the algorithm are of polynomial-time, and can retrieve invariants from arbitrarily high dimensional QP-ODEs with arbitrarily high number of quasi-monomials.

Both versions of the algorithm has been implemented in the MATLAB numeric computational environment, and their operation has been tested successfully on the mathematical model of several physical systems.

The invariance properties of the algorithm have been investigated under quasi-monomial and algebraic equivalence transformations. The capabilities and limitations of the algorithm have also been discussed, and compared with another known method, the QPSI method [17]. According to its applicability, simplicity and effectiveness even in cases of high-dimensional models with arbitrary high number of quasi-monomials, the proposed algorithm is proved to be a feasible alternative of the QPSI approach.

4   Application areas, directions of future research

Since a detailed nonlinear model of a real low-power gas turbine has been used for the stability analysis and controller design thereon, the implementation of the controllers designed, especially the one that estimates the load torque on an experimental test-bed (on a real turbine) is the first step towards the application of theoretical results. Then, this novel controller should be implemented to other gas turbines.

There are a lot of possible directions of future research connected to the analysis and control of the low-power gas turbine:

- For the stability analysis of the zero dynamics for the turbine inlet total pressure, another method could also be applied, e.g. *bilinear* matrix inequalities, being hopefully less conservative than the method based on LMIs.

- Although the one dimensional zero dynamics of the gas turbine for the rotational speed has a very complicated algebraic structure, its phase diagram is a relatively simple curve. The identification of the phase curve as a polynomial depending on two constant parameters (rotational speed and load torque) might give the possibility of the analytical treatment of stability investigation.

- The robustness of the controllers could only be checked via simulations, however an analytical treatment of uncertainties would be desirable. Further work should be focused on obtaining a structured description of the key parametric and dynamic uncertainties of the gas turbine model that would guarantee better circumstances for robust controller design.

Since the majority of classical nonlinear input-affine models can be transformed to a QP-ODE, the algorithm for retrieving invariants could be an effective and easily applicable tool to determine QP-type first integrals.

The possible directions of future work related to the topic are as follows:

- The algorithm should be re-implemented in a symbolic computer-algebra system (e.g. MAPLE) in order to handle parametric models.

- Moreover, this algorithm should be applied constructively in control oriented nonlinear system analysis and feedback design - for instance by combining it with control Lyapunov function based techniques.

5 Publications related to the theses

[P1] B. Pongrácz, G. Szederkényi, P. Ailer, and K. M. Hangos. Stability of zero dynamics of a low-power gas turbine. In *Proceedings of the 12th Mediterranean Control Conference - MED'04,* Turkey, 2004. *On CD.* (**Thesis 1**)

[P2] B. Pongrácz. Nonlinear stability analysis and control of a low power gas turbine. In *Proceedings of the 2nd PhD Mini-Symposium,* pages 60-62, Veszprém, Hungary, 2004. (**Theses 1,2**)

[P3] B. Pongrácz, P. Ailer, G. Szederkényi, and K. M. Hangos. Nonlinear zero dynamics analysis and control of a low power gas turbine. In *Proceedings of the 5th PhD Workshop on Systems and Control - a Young Generation Viewpoint,* pages 112-116, Balatonfüred, Hungary, 2004. (**Theses 1,2**)

[P4] B. Pongrácz, P. Ailer, G. Szederkényi, and K. M. Hangos. Nonlinear zero dynamics analysis and control of a low power gas turbine. *Technical report SCL-003/2005, Computer and Automation Research Institute, HAS*, 2005.
`http://daedalus.scl.sztaki.hu/pdf/research reports/`
`SCL-003-2005.pdf` (**Theses 1,2**)

[P5] P. Ailer, B. Pongrácz, and G. Szederkényi. Constrained control of a low power industrial gas turbine based on input-output linearization. In *Proceedings of the International Conference on Control and Automation - ICCA'05,* 2005. *On CD.*
(**Thesis 2**)

[P6] B. Pongrácz, P. Ailer, G. Szederkényi, and K. M. Hangos. Nonlinear reference tracking control of a gas turbine with load torque estimation. *International Journal of Adaptive Control and Signal Processing,* in print. DOI: 10.1002/acs.1020 (**Thesis 2**)
Impact factor: 0.580

[P7] A. Magyar, G. Ingram, B. Pongrácz, G. Szederkényi, and K. M. Hangos. On some properties of quasi-polynomial differential equations and differential-algebraic equations. *Technical report SCL-010/2003, Computer and Automation Research Institute, HAS*, 2003.
`http://daedalus.scl.sztaki.hu/PCRG/works/SCL-010-2003.pdf`
(**Thesis 3**)

[P8] B. Pongrácz, G. Ingram, and K. M. Hangos. The structure and analysis of QP-DAE system models. *Technical report SCL-004/2004, Computer and Automation Research Institute, HAS*, 2004.
`http://daedalus.scl.sztaki.hu/pdf/research reports/`
`SCL-004-2004.pdf` (**Thesis 3**)

[P9] B. Pongrácz, G. Szederkényi, and K. M. Hangos. An algorithm for determining invariants in quasi-polynomial systems. In *Proceedings of the 6th PhD Workshop on Systems and Control - a Young Generation Viewpoint,* Izola, Slovenia, 2005. *On CD.*
(**Thesis 3**)

[P10] B. Pongrácz, G. Szederkényi, and K. M. Hangos. An algorithm for determining a class of invariants in quasi-polynomial systems. *Technical report SCL-002/2005, Computer and Automation Research Institute, HAS*, 2005.
`http://daedalus.scl.sztaki.hu/pdf/research reports/`
`SCL-002-2005.pdf` (**Thesis 3**)

[P11] B. Pongrácz, G. Szederkényi, and K. M. Hangos. An algorithm for determining a class of invariants in quasi-polynomial systems. *Computer Physics Communications,* **175:**204-211, 2006. DOI: 10.1016/j.cpc.2006.03.003 (**Thesis 3**)
Impact factor: 1.595

6 Publications not related to the theses

[O1] B. Pongrácz. An algorithm for transforming a class of DAE models into a purely differential form. In *Proceedings of the 2nd PhD Workshop on Systems and Control - a Young Generation Viewpoint,* pages 31-40, Balatonfüred, Hungary, 2001.

[O2] B. Pongrácz, G. Szederkényi, and K. M. Hangos. The effect of algebraic equations on the stability of process systems. In *Proceedings of the 3rd International PhD Workshop on Systems and Control*, Strunjan, Slovenia, 2002. *On CD.*

[O3] B. Pongrácz, G. Szederkényi, and K. M. Hangos. The effect of algebraic equations on the stability of process systems. *Technical Report, SCL-003/2002 Computer and Automation Research Institute, HAS*, 2002.
`http://daedalus.scl.sztaki.hu/PCRG/works/SCL-003-2002.pdf`

[O4] B. Pongrácz. Stability analysis techniques for process models in DAE form - The effect of algebraic equations. In *Proceedings of the 1st PhD Mini-Symposium*, pages 55-57, Veszprém, Hungary, 2003.

[O5] B. Pongrácz, G. Szederkényi, and K. M. Hangos. The effect of algebraic equations on the stability of process systems modelled by differential algebraic equations. In *Proceedings of the 13th European Symposium on Computer Aided Process Engineering - ESCAPE-13,* pages 857-862, Finland, 2003.

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