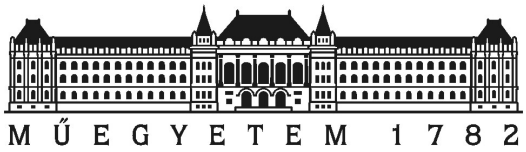

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS
FACULTY OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICAL ANALYSIS

Investigation of the Extension Possibilities of the Martingale Quantum State Estimation Method for Two-Level Quantum Systems

Master's Thesis

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Budapest, 2013

Declaration

I, the undersigned Krisztián Balázs, do hereby declare that the work contained in this thesis for the purpose of obtaining my Master of Science degree in Mathematics is my original work and that I have not used any other than those listed in the bibliography and cited in the text. Each part which has been taken explicitly, or with similar content from other sources has been referred with the explicit source.

Budapest, May 24, 2013

Krisztián Balázs

Nyilatkozat

Alulírott, Balázs Krisztián ezennel kijelentem, hogy ezt a diplomamunkát Matematikus diplomám megszerzése céljából magam készítettem, és abban csak az irodalomjegyzékben feltüntetett és a szövegben hivatkozott forrásokat használtam fel. Minden olyan részt, amelyet szó szerint, vagy azonos tartalommal, de átfogalmazva más forrásból átvettem, egyértelműen, a forrás megadásával megjelöltem.

Budapest, 2013. május 24.

Balázs Krisztián

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Chapter 1

Introduction

1.1 Background and motivation

Before the late 19th century it was a widely accepted consideration among researchers and practitioners of physics that physics as a scientific field had been almost completely discovered and that there were only some minor unanswered questions. These questions included the description of e.g. thermal radiation and photoelectronic effect as well as dealing with the inconsistencies in aether models [18]. However, the theories of physics at that time (since then referred to as *classical physics*) seemed to be incapable of resolving these questions, which was a very disturbing recognition leading to the development of new, revolutionary theories and to paradigm shifts in physics around the turn of the 20th century for explaining the mentioned phenomena. With these new theories a new branch of science, namely *modern physics*, was born.

Basically, the areas of modern physics can be divided into two overlapping parts. The first part is *quantum theory* providing tools and models for describing nature in small scales, whereas the second part is *relativity theory* attempting to explain the phenomena arising in large scales and besides high velocities [18]. Theories combining these two parts can also be found in modern physics.

Quantum mechanics is a mathematical framework serving as a fundamental and inseparable component of quantum theory. Due to the developments achieved based on quantum mechanics, a number of physical systems became examinable and even controllable at the atomic scale. However, quantum mechanics not only establishes an environment for models in quantum physics, but also provides the ground of *quantum computation* and *quantum information theory* [7].

Quantum computation is an intensively researched field of computer science. Its purpose is to adopt the models and results of classical computer science into the frame of quantum mechanics, and to improve them by exploiting the new possibilities arising in the quantum mechanical environment. The abstract, mathematical concepts of quantum computers have already been developed, however, efficient physical realizations of such machines have not been invented up to the present.

Analogously to bits in case of classical computers, in quantum computers *quantum bit*s serve as the basic units of information. Hence, quantum bits play a fundamental role in

the theory of quantum computation. Quantum bits (also known as *qubits*) are *two-level quantum system*s and thus — as it will be obvious after the first section of Chapter 2 — their possible states significantly differ from the states of classical bits, hence qubits can represent much more information than classical bits. A consequence of this fact is that when quantum phenomena were attempted to be simulated on classical computer architectures, it was realized that, although, classical computers could be programmed for such simulations, they were totally inappropriate for this task due to the lack of performance originating from the bounded state representation possibilities, however, quantum computers would be able to implement these simulations efficiently in a straightforward way [7].

Nevertheless, quantum computers would not only be efficient in simulating quantum phenomena. They would be able to achieve significant improvements in everyday tasks being solved nowadays by classical computers. These improvements include algorithms for searching, ordering and prime factorization with lower computational complexity, more efficient data compression and communication methods as well as much safer cryptography techniques, to mention only a few [7].

1.2 Problem statement and aims

Two-level quantum systems (or quantum bits) are the simplest nontrivial quantum systems. Since in quantum computation and quantum information theory they serve as the basic units of information, their efficient measurement and estimation, which means the 'reading of the stored information', is an essential and important problem. The measurement of quantum bits is far not as easy as in case of classical bits, because on one hand, a single measurement of a quantum system can extract only a 'portion' of the state, and on the other hand, only one single measurement may affect the state drastically in an irreversible way [7].

For this reason many approaches have been proposed for estimating the states of quantum systems, especially two-level quantum systems (see e.g. [10], [9], [11] and [12]). Recently a *Martingale based state estimation method* has been developed, which has shown very promising properties from the point of view of estimation efficiency [15]. However, the drawback of this technique is that it is capable of estimating only one single dimension of the state of a two-level quantum system. In order to estimate multiple dimensions of the state, the estimation process has to be repeated for each dimension separately.

The aim of my master's thesis work was to generalize the Martingale method by making it capable of estimating multiple dimensions of the state of a given two-level quantum system simultaneously, in the case when the state alters during the estimation process only due to interacting with the measurement system, i.e. to extend the Martingale technique to multiple dimensions, when — with the notions of Chapter 2 — there is no local time evolution in the measured system. The extension had to deal with at least two-dimensions, but the ultimate goal was to estimate the whole state of the given two-level quantum system simultaneously.

1.3 Methodology

The extension of the Martingale method to multiple dimensions requires the solutions of equations derived from the dynamics of coupled quantum systems. In other words, mathematical expressions having nonlinear, trigonometric terms have to be appropriately parameterized.

Due to this complicated nature of the task that my master's thesis aimed to solve, in the first part of my work symbolic and numerical software computations were applied on a personal computer. For this purpose, as the first step, a simulation environment was established in Matlab (standing for "Matrix laboratory") [6]. Matlab is a software framework being a very efficient tool for computing with matrices, as its name suggests. Then, using the implemented environment the proper general equations for parameterization were derived. After that, numerical algorithms were applied in order to find appropriate values for the parameters.

Since the results achieved by the symbolic and numerical computations led to the formulation of strange, previously unexpected conjectures, in the second part of my work the problem was investigated analytically and, as a result, theorems were formulated and proved determining whether the conjectures made were correct or not.

Throughout the thesis it is assumed that the reader is familiar with *linear algebra* [5], *functional analysis* [14], *abstract algebra* [1] and *probability theory* [3] on a graduate level of Mathematics.

1.4 Thesis structure

At this point, at the end of the introductory chapter, the organization of this thesis is described.

This first chapter, "Introduction", discusses the context of my research starting with a short description of the scientific field in question and the motivations for studying it. Then, open problems and research directions are highlighted before declaring the objectives of my research. After describing the aims, the methodology for accomplishing them is explained. This chapter ends with presenting the structure of the thesis.

The second chapter, "Theoretical overview", briefly describes the theoretical background of my work, which is necessary in order to understand the new results I achieved. The first section of the chapter explains the basics of quantum mechanics in a nutshell focusing on the notions applied later both in the remaining part of the introduction and in the explanation of the new results. After this, the decomposition of the time evolution of coupled two-level quantum systems is described. The third part of the chapter introduces indirect measurements, where both the general concept and the Martingale method are explained.

The third chapter, "Simulation based investigation of the extension possibilities of the Martingale method", describes the results I achieved using symbolic and numerical computations in the simulation environment I implemented in Matlab. The analysis in this chapter followed two significantly different paths confirming each other. The most important formulea derived by the symbolic computations, graphical illustrations of the results and the consequences of the analysis are also presented in the descriptions.

The fourth chapter, "Analytical characterization of indirect measurement cycles of two-level quantum systems", carries out an analytical investigation motivated by the unexpected results of the previous chapter. This analysis formulates and proves statements about indirectly measured two-level quantum systems. The chapter ends with two theorems regarding the results of Chapter 3.

The last chapter, "Conclusions", summarizes the thesis, draws some conclusions on my research work and outlines possible objectives for further research.

This is followed by the "Bibliography", which enumerates the referred literature.

Chapter 2

Theoretical overview

The aim of this chapter is to concisely present the bases of those areas of quantum theory that served as the ground of my work. Its purpose is to place my research into context and to clarify its initial state. Although, even a basic discussion of these areas could easily exceed the size of this thesis, after the brief introduction given in this chapter both the ideas and the details of my new results as well as their contributions become understandable and acquirable.

The first section of the chapter deals with the basic concepts of quantum mechanics and two-level quantum systems. Then, a decomposition method of the dynamics of coupled two-level quantum systems is described. The third section introduces the general concept of indirect measurement, explains its application for two-level systems and describes the so-called Martingale method.

2.1 Quantum mechanical bases

Quantum mechanics is a mathematical framework [7], with the help of which quantum systems and quantum phenomena may be modeled appropriately. The states and behaviors of quantum systems can be described based on the *quantum mechanical postulates*, which are considered to be the fundamental building blocks of quantum theory.

2.1.1 Postulates of quantum mechanics

The postulates of quantum mechanics have different equivalent forms in the literature [7]. Each form has exactly the same expression power, however, in different circumstances different forms provide easier usability. Due to the areas touched by my research, during my work it was reasonable to use the following forms [7].

> **Postulate 1.** To every closed physical system a *state space* can be assigned, which is a separable complex Hilbert-space [14]. The system can be completely represented by its *density operator* (or *density matrix*) being a trace-one positive semidefinite operator acting on the state space. If a quantum system is in a state characterized by the density operator ρ_i with probability p_i, then its density matrix is equal to $\sum_i p_i \rho_i$.

Postulate 2. The *time evolution* of every closed quantum system, i.e. the series of states the system passes through, can be described by a two-parameter $\{U(t_1, t_2)\}$ unitary operator group. From state $\rho(t_1)$ in time t_1 the state $\rho(t_2)$ in time t_2 can be obtained using the following equation:

$$\rho(t_2) = U(t_1, t_2)\rho(t_1)U(t_1, t_2)^*, \tag{2.1}$$

where * denotes the adjoint (conjugate transpose) of the specified matrix.

Postulate 3. The *measurement* of quantum systems, i.e. the examination of their states, can be modeled by the application of the measurement operators $\{M_m\}$ acting on the state space of the system being measured and fulfilling the equation $\sum_m M_m^* M_m = \mathrm{I}$. The index m of a particular measurement operator corresponds to the *outcome* of the measurement. If the state of a quantum system is ρ before the measurement, then the probability of outcome m is

$$\mathrm{Tr}(M_m \rho M_m^*), \tag{2.2}$$

where Tr and * denote the trace and the adjoint (conjugate transpose) operators, respectively, and the state of the system after the measurement becomes

$$\frac{M_m \rho M_m^*}{\mathrm{Tr}(M_m \rho M_m^*)}. \tag{2.3}$$

Postulate 4. The state space of *composite* (*compound*) quantum systems can be represented by the tensor product of the component systems. Furthermore, considering a compound system with n components, where the states of the components are described by the density matrices $\{\rho_i\}_{i=1}^n$, then the total state of the compound system is equal to $\rho_i \otimes \rho_2 \otimes \cdots \otimes \rho_n$.

2.1.2 Pauli matrices

The *Pauli matrices* have a fundamental role in the theory of two-level quantum systems due to the well-known fact [14] that every self-adjoint matrix A of type 2×2 can be represented in the basis of the Pauli matrices as follows:

$$A = \sum_{i=0}^{3} \lambda_i \sigma_i, \tag{2.4}$$

where $\{\sigma_i\}_{i=0}^3$ are the Pauli matrices:

$$\sigma_0 = \mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.5}$$

After some calculations Table 2.1 shows the products of different Pauli matrices. Since matrix multiplication is associative, based on this table it is straightforward to see that the identity matrix, the three other Pauli matrices multiplied by the imaginary unit and the additive inverses of these matrices together with matrix multiplication form a group $(\{\pm \mathrm{I}, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3\}, \cdot)$ being isomorphic to the Quaternion group with correspondences $1 \leftrightarrow \mathrm{I}, i \leftrightarrow i\sigma_3, j \leftrightarrow i\sigma_2, k \leftrightarrow i\sigma_1$.

Table 2.1: *Products of Pauli matrices*

	σ_0	σ_1	σ_2	σ_3
σ_0	σ_0	σ_1	σ_2	σ_3
σ_1	σ_1	σ_0	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	σ_0	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	σ_0

2.1.3 Bloch vector representation

*Two-level quantum system*s, i.e. those systems, whose state space is $\mathbb{C}^2$, are the *quantum bits*, or alternatively *qubits*. According to Postulate 1 the states of qubits can be described by density operators of type 2×2. As it was mentioned above, every self-adjoint matrix A of type 2×2 can be represented in the basis of the Pauli matrices (cf. Eq. 2.4).

Since density operators are trace-one positive semidefinite matrices, it can be shown [8] that a matrix A is a density matrix if and only if for the coefficients of its representation in the Pauli basis $\lambda_0 = \frac{1}{2}$ and $\sum_{i=1}^{3} 4\lambda_i^2 \leq 1$. For technical purposes the coefficients λ_i are commonly substituted by the coefficients $x_i := 2\lambda_i$ ($i \in \{0,1,2,3\}$). Thus, a density operator ρ has the following form:

$$\rho = \frac{1}{2}(\mathrm{I} + x \cdot \sigma), \text{ where} \tag{2.6}$$

$$x = (x_1, x_2, x_3), \sigma = (\sigma_1, \sigma_2, \sigma_3).$$

This way a representation is obtained, where the state is characterized by the *Bloch vector* x within the 3-dimensional closed unit ball, the *Bloch ball*.

Since the Pauli matrices are orthogonal to each other, coordinate $k \in \{1,2,3\}$ of the Bloch vector can be extracted from the density matrix ρ by calculating the inner product

$$\mathrm{Tr}(\rho^* \sigma_k) = \mathrm{Tr}(\rho \sigma_k) = \mathrm{Tr}\left[\frac{1}{2}\left(\mathrm{I} + \sum_{i=1}^{3} x_i \sigma_i\right)\sigma_k\right] = x_k. \tag{2.7}$$

2.1.4 Entangled states

Consider a composite system established from two two-level component quantum systems. Assume that the components have the same initial states

$$\rho_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \rho_A. \tag{2.8}$$

Then, according to Postulate 4 the initial state of the total system is

$$\rho_T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.9}$$

Assume that on the first system component a time evolution described by the *Hadamard transform*

$$U_H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.10}$$

is applied followed by the *controlled not* transform

$$U_{cn} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.11}$$

on the whole system — which operations are by the way fundamental building blocks in the theory of quantum computation [7].

Then, the resulting state is a so-called *Bell state* [7] — also having an important role in quantum computation — that can be expressed using Postulate 2 as follows:

$$\rho_{Bell} = U_{cn}(U_H \otimes \mathrm{I})(\rho_S \otimes \rho_A)(U_H \otimes \mathrm{I})^* U_{cn}^* = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} \sum_{i=1}^{4} E_i \otimes E_i, \tag{2.12}$$

where

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.13}$$

However, the obtained Bell state is not in the space spanned by the tensor products of second order quadratic complex matrices, i.e. $\rho_{Bell} \notin M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, otherwise since $\{E_i\}_{i=1}^4$ is a basis of $M(\mathbb{C})_2$,

$$\rho_{Bell} = \left(\sum_{i=1}^{4} \alpha_{i,S} E_i \right) \otimes \left(\sum_{j=1}^{4} \alpha_{j,A} E_j \right) = \sum_{i=1}^{4} \sum_{j=1}^{4} \alpha_{i,S} \alpha_{j,A} E_i \otimes E_j \tag{2.14}$$

would hold for appropriate coefficients $\alpha_{i,S}, \alpha_{j,A} \in \mathbb{C}$ $(i, j \in \{1, 2, 3, 4\})$, and then

$$\forall i \neq j \colon \ \alpha_{i,S} \alpha_{j,A} = 0 \tag{2.15}$$

would follow, consequently, at most one of the products $\{\alpha_{i,S}\alpha_{i,A}\}_{i=1}^4$ could be nonzero, which would be a contradiction, since all these products must be equal to $\frac{1}{2}$ according to Eq. 2.12.

Besides the fact that the relation $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \subsetneq M_4(\mathbb{C})$ holds, this example shows more. Namely, that it is easy to find initial states and unitary operators, for which after time evolution the state of the composite system cannot be expressed by the tensor product of component states.

This phenomenon can also be observed in more general cases, when either the number of the component systems is greater or the state spaces of the components are higher dimensional Hilbert-spaces.

Such composite states that cannot be expressed by the tensor product of component states are the *entangled state*s.

2.1.5 Partial trace

Postulate 4 describes the way for computing the total state of a composite quantum system based on the states of its components. The reverse method also becomes of interest when the total state is known and the states of particular components are inquired.

For this purpose the *partial trace* operator can be introduced [7]. Before defining this operator, it should be noted that in case of entangled states, no exact reverse method may exist, in the sense that whatever component states are obtained by any operations, their tensor product will certainly not be the entangled state by definition.

Assume a composite quantum system established from two components, where the components are two-level quantum systems in states ρ_1 and ρ_2. Then, the total state is

$$\rho_T = \rho_1 \otimes \rho_2 := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix} \tag{2.16}$$

The partial traces for the first and the second components, Tr_1 and Tr_2, are meant to extract the second and the first component states, ρ_2 and ρ_1, respectively, from the composite state. Hence the definitions of the partial traces are

$$\mathrm{Tr}_1(\rho_T) := \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} a_{3,3} & a_{3,4} \\ a_{4,3} & a_{4,4} \end{bmatrix}, \tag{2.17}$$

$$\mathrm{Tr}_2(\rho_T) := \begin{bmatrix} \mathrm{Tr}\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} & \mathrm{Tr}\begin{bmatrix} a_{1,3} & a_{1,4} \\ a_{2,3} & a_{2,4} \end{bmatrix} \\ \mathrm{Tr}\begin{bmatrix} a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \end{bmatrix} & \mathrm{Tr}\begin{bmatrix} a_{3,3} & a_{3,4} \\ a_{4,3} & a_{4,4} \end{bmatrix} \end{bmatrix}. \tag{2.18}$$

It is easy to see that if the total state is not entangled, then the partial trace always extracts the appropriate component states from the tensor product.

The fact is worth mentioning that for arbitrary tensor product $A \otimes B$ the following holds true:

$$\mathrm{Tr}_1(A \otimes B) = B \cdot \mathrm{Tr}(A), \tag{2.19}$$

$$\mathrm{Tr}_2(A \otimes B) = A \cdot \mathrm{Tr}(B). \tag{2.20}$$

It can also be verified that the application of the partial trace operator on a composite state always results in a valid state, even if the composite state is entangled.

Although, the partial trace operator can be generalized in a straightforward way to composite systems having more components being more general, this generalization is omitted here, because in this thesis only the above described special case will be used.

2.2 Decomposition of the time evolution of coupled two-level quantum systems

Postulate 2 and Postulate 3 clearly describe the dynamics of a simple two-level quantum system during time evolution and measurement. It is easy to verify that for every quantum

state the application of arbitrary possible time evolution and measurement results in a valid quantum state.

However, when two two-level quantum systems are attached (*coupled*) to each other, the description is more difficult as entangled states arise. Although, it is still true that the postulates describe the dynamics of the whole system, the components can no longer be analyzed individually.

Fortunately, a decomposition method has been invented [13] easing the difficulty of the description of the time evolution of coupled two-level quantum systems.

Based on representation theoretical considerations [2], using Cartan decomposition (see e.g. [4]), every unitary operator $U \in SU(4)$ acting on the composite system can be decomposed in the following way [13]:

$$U = (L_{1,1} \otimes L_{1,2}) e^{at} (L_{2,1} \otimes L_{2,2}), \tag{2.21}$$

where $L_{1,1}, L_{1,2}, L_{2,1}, L_{2,2} \in SU(2)$ express the *local time evolutions* of the component systems and $a = i(c_1\sigma_1 \otimes \sigma_1 + c_2\sigma_2 \otimes \sigma_2 + c_3\sigma_3 \otimes \sigma_3)$ for appropriate $c_1, c_2, c_3 \in \mathbb{R}$ values forming a real parameter vector

$$c := (c_1, c_2, c_3) \in \mathbb{R}^3. \tag{2.22}$$

(In the above expressions $SU(n)$ denotes the special unitary group of order n.)

Based on Table 2.1 the products of matrices $\sigma_i \otimes \sigma_i \, (i \in \{0,1,2,3\})$ are presented in Table 2.2. Considering that matrix multiplication is associative, this table shows that $(\{\pm\sigma_0 \otimes \sigma_0, \pm\sigma_1 \otimes \sigma_1, \pm\sigma_2 \otimes \sigma_2, \pm\sigma_3 \otimes \sigma_3\}, \cdot)$ is a group, where the operator '$\cdot$' is matrix multiplication.

Table 2.2: *Products of matrices* $\sigma_i \otimes \sigma_i \, (i \in \{0,1,2,3\})$

	$\sigma_0 \otimes \sigma_0$	$\sigma_1 \otimes \sigma_1$	$\sigma_2 \otimes \sigma_2$	$\sigma_3 \otimes \sigma_3$
$\sigma_0 \otimes \sigma_0$	$\sigma_0 \otimes \sigma_0$	$\sigma_1 \otimes \sigma_1$	$\sigma_2 \otimes \sigma_2$	$\sigma_3 \otimes \sigma_3$
$\sigma_1 \otimes \sigma_1$	$\sigma_1 \otimes \sigma_1$	$\sigma_0 \otimes \sigma_0$	$-\sigma_3 \otimes \sigma_3$	$-\sigma_2 \otimes \sigma_2$
$\sigma_2 \otimes \sigma_2$	$\sigma_2 \otimes \sigma_2$	$-\sigma_3 \otimes \sigma_3$	$\sigma_0 \otimes \sigma_0$	$-\sigma_1 \otimes \sigma_1$
$\sigma_3 \otimes \sigma_3$	$\sigma_3 \otimes \sigma_3$	$-\sigma_2 \otimes \sigma_2$	$-\sigma_1 \otimes \sigma_1$	$\sigma_0 \otimes \sigma_0$

Using this group property and the Taylor-series of the exponential function it can be proved [13] that

$$e^{at} = \sum_{i=0}^{3} \mu_i(c,t)\sigma_i \otimes \sigma_i, \tag{2.23}$$

where

$$\mu_0(c,t) = \cos(c_1 t)\cos(c_2 t)\cos(c_3 t) + i\sin(c_1 t)\sin(c_2 t)\sin(c_3 t), \tag{2.24}$$

$$\mu_1(c,t) = \cos(c_1 t)\sin(c_2 t)\sin(c_3 t) + i\sin(c_1 t)\cos(c_2 t)\cos(c_3 t), \tag{2.25}$$

$$\mu_2(c,t) = \sin(c_1 t)\cos(c_2 t)\sin(c_3 t) + i\cos(c_1 t)\sin(c_2 t)\cos(c_3 t), \tag{2.26}$$

$$\mu_3(c,t) = \sin(c_1 t)\sin(c_2 t)\cos(c_3 t) + i\cos(c_1 t)\cos(c_2 t)\sin(c_3 t). \tag{2.27}$$

Thus, by substituting Eq. 2.23 into Eq. 2.21 and using the properties of the tensor product, the decomposition of $U \in SU(4)$ becomes

$$U = \sum_{i=0}^{3} \mu_i(c,t) L_{1,1}\sigma_i L_{2,1} \otimes L_{1,2}\sigma_i L_{2,2}. \tag{2.28}$$

2.3 Indirect measurement

In contrast with time evolution, during the measurement of a quantum system the state of the system is strongly affected by the measurement procedure. Consequently, as it can be clearly seen from Postulate 3, the state may alter drastically in an irreversible way. This phenomenon is the *measurement back-action* [7].

In order to avoid this unfavorable back-action *indirect measurement* can be applied.

2.3.1 General concept

Generally speaking, when indirect measurement is performed, instead of examining the original quantum system, an auxiliary or *ancilla* system coupled to the original one is measured iteratively. In each iteration the following steps take place [17]:

1. An ancilla system possessing a known initial state is coupled with parameters $c = (c_1, c_2, c_3)$ (see Eq. 2.22) to the system to be measured.

2. The state of the resulting compound quantum system evolves according to Postulate 2 and Postulate 4 (by unitary evolution described in Eq. 2.28) during a predefined time period t.

3. Finally, the ancilla system is measured directly and the two systems are decoupled. The decoupling of the systems is modeled by applying the partial trace operator (see Subsection 2.1.5) on the ancilla system.

In Step 3 the probability distribution of the outcome of the measurement performed on the ancilla system is determined by the known initial state in Step 1 and the time evolution in Step 2 depending on the initial state of the measured system. Hence, considering the initial state of the ancilla system, the coupling parameters c and the length of the time evolution t, and based on the statistics of the outcomes of the indirect measurement during the iterations, the original state of the measured quantum system can be estimated.

2.3.2 Martingale method for the indirect measurement of two-level quantum systems

In the remaining part of this section the indirect measurement of two-level quantum systems will be discussed using the Bloch vector representation (see Subsection 2.1.3).

Certain coordinates of the qubits (x_1, x_2 and x_3) can be measured by the respective projections of the spectral decompositions of the $\{\sigma_i\}_{i=1}^3$ Pauli matrices. These projections play the role of the measurement operators in Postulate 3. As it can be easily verified, the eigenvalues of each Pauli matrix are $+1$ and -1, furthermore the projections in the spectral decomposition of $\sigma_i = (+1)E_{+1}^{(i)} + (-1)E_{-1}^{(i)}$ ($i \in \{1,2,3\}$) are the following:

$$E_{\pm 1}^{(i)} = \frac{1}{2}(\mathrm{I} \pm \sigma_i). \tag{2.29}$$

Based on Postulate 3, Eq. 2.29 and Table 2.1, the outcome of the measurements along coordinate i can take values ± 1 with probabilities

$$\begin{aligned}\mathbb{P}(\pm 1) = \mathrm{Tr}\left(\rho E_{\pm 1}^{(i)}\right) = \frac{1}{4}\,\mathrm{Tr}\left((\mathrm{I} + x \cdot \sigma)(\mathrm{I} \pm \sigma_i)\right) = \\ = \frac{1}{4}\left(\mathrm{Tr}(\mathrm{I}) \pm \mathrm{Tr}(x_i \cdot \mathrm{I})\right) = \frac{1 \pm x_i}{2},\end{aligned} \tag{2.30}$$

because the three Pauli matrices $\{\sigma_i\}_{i=1}^3$ are traceless.

Denote by $x_i(k)$ ($i \in \{1,2,3\}$) a particular coordinate (first, second or third) of the Bloch vector $x(k)$ of the measured system in iteration k of the indirect measurement. Consider the special case, when [16]

- there is no local time evolution in the component systems, i.e. in Eq. 2.28

$$L_{1,1} = L_{1,2} = \mathrm{I} = L_{2,1} = L_{2,2}, \tag{2.31}$$

- for the parameters of the coupling $c_i t = \frac{-\pi}{4}$ and if $j \neq i$, then $c_j = 0$ hold true,
- the measurement operators are the projections of the spectral decomposition of $(\mathrm{I} \otimes \sigma_j)$, where j takes the values $1, 2$ and 3 exactly when the value of i is $2, 3$ and 1, respectively.

Then, it can be shown [17] that the state in the subsequent iteration depending on the outcome given by the ancilla system is

$$x_i(k+1) = \begin{cases} \frac{x_i(k)+y_j(k)}{1+x_i(k)y_j(k)}, & \text{with probability } \frac{1+x_i(k)y_j(k)}{2} \quad \text{(outcome } +1\text{)} \\ \frac{x_i(k)-y_j(k)}{1-x_i(k)y_j(k)}, & \text{with probability } \frac{1-x_i(k)y_j(k)}{2} \quad \text{(outcome } -1\text{)} \end{cases}, \tag{2.32}$$

where j takes the values $1, 2$ and 3 exactly when the value of i is $3, 1$ and 2, respectively, and $y_j(k)$ is the corresponding coordinate of the ancilla state. Hereafter, assume that in each iteration identical ancilla qubits are used, i.e. the ancilla state is the same at the beginning of every iteration, $y_j(k) \equiv y_j$.

It can be proved [16] that if the difference d between the number of $+1$ and -1 outcomes is the same in two different measurement series, then the final values of x_i after the whole measurement process are equal. Therefore, the value of coordinate i can be characterized by its initial value and this difference d. Henceforth, these states will be denoted by x_d, where

$x_0 := x_i(0)$. In further notations coordinate i will not appear in the subscripts, however, bear in mind that the formulae below correspond to only the selected coordinate i.

It is rather straightforward to verify that if the state is expressed in the form $x_d = \frac{z_d}{q_d}$ and if the inequality $d > 0$ holds true, then the values of x_d are obtained by the recursion [15]

$$(z_{d+1}, q_{d+1}) = (z_d + y_j q_d, q_d + y_j z_d), \tag{2.33}$$

where $z_0 = x_0$ and $q_0 = 1$. It can be easily seen that z_d and q_d are linear in terms of x_0, moreover, if $z_d = a_d + b_d x_0$, then $q_d = b_d + a_d x_0$, where a_d and b_d are polynomials of y_j. Utilizing these facts:

$$x_d = \frac{a_d + b_d x_0}{b_d + a_d x_0}. \tag{2.34}$$

Similarly, it can be obtained that

$$x_{-d} = \frac{-a_d + b_d x_0}{b_d - a_d x_0}, \tag{2.35}$$

where a_d and b_d are the very same as above.

According to this discussion the state estimation of quantum bits involving indirect measurement can be reduced to the task of estimating x_0 on the basis of the ancilla state coordinate y_j and the outcomes of the measurements.

By substitution it can be easily verified that the above defined stochastic process $x_i(k)$ is a martingale, for which Doob's Optional Stopping Theorem [3] is applicable with the stopping time $\tau = \inf\{k : |d(k)| = D\}$, where $d(k)$ is the difference between the number of outcomes $+1$ and -1 in terms of the number of executed measurement iterations and $D \in \mathbb{Z}^+$ is a predefined parameter. Due to Doob's theorem if $p_+ = \mathbb{P}(x(\tau) = +D)$, then the following equation must hold true:

$$x_0 = \mathbb{E}x_0 = \mathbb{E}x(0) = \mathbb{E}x(\tau) = \mathbb{E}x_{d(\tau)} = p_+ x_{+D} + (1 - p_+)x_{-D}. \tag{2.36}$$

Rearranging this and substituting the corresponding variables of Eq. 2.34 and Eq. 2.35:

$$p_+ = \frac{x_0 + x_{-D}}{x_{+D} + x_{-D}} = \frac{1}{2}\left(1 + \frac{a_D}{b_D} x_0\right). \tag{2.37}$$

Expressing the measured coordinate:

$$x_0 = (2p_+ - 1)\frac{b_D}{a_D}. \tag{2.38}$$

On the basis of the Law of Large Numbers [3], if measurement series are performed repeatedly — on different copies of the measured qubit — until the difference between the number of outcomes $+1$ and -1 reaches $+D$ or $-D$, then the ratio of the number of measurement series (N_+), where $+D$ was reached, and the number of all measurement series (N), i.e. the number of qubits consumed, tends to p_+ as N tends to infinity. Since p_+ is linear in terms of x_0 (see Eq. 2.37), an unbiased $\hat{x}_0$ estimation is obtained for the measured coordinate:

$$\hat{x}_0 = \left(\frac{2N_+}{N} - 1\right)\frac{b_D}{a_D} \to x_0, \text{ as } N \to \infty. \tag{2.39}$$

2.3.3 A brief discussion of the Martingale method

Based on both theoretical and experimental results the efficiency of the Martingale method compared to other indirect state estimation techniques (Conditional histogram [15] and Bayes approaches [15]) were discussed in [15]. According to this discussion, where the methods were confronted with each other from several aspects, the Martingale technique appeared to be the most efficient one.

Using computer simulations the performance of the Martingale approach during the adjustment of its parameters y_j and D were also analyzed in [15].

A similar experiment is shown in Figure 2.1 illustrating the above mentioned unbiasedness, where the Mean Squared Error (MSE) values are presented in terms of consumed qubits during two different (squares and circles) indirect measurement series carried out in a computer simulation environment. The squares and the circles denote the results in case of parameter values $D = 10$ and $D = 100$, respectively.

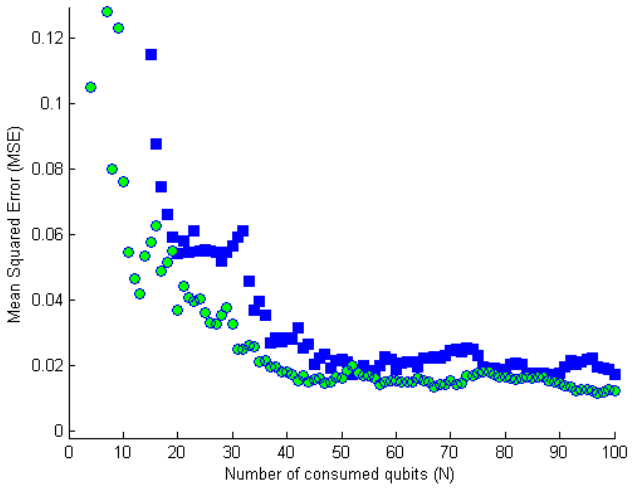

Figure 2.1: *Mean Squared Error (MSE) values in terms of the number of consumed qubits (N).*

Finally, the potential generalization possibilities were also mentioned in [15], namely, the idea of finding parameterizations of the indirect measurement setup, for which the technique could be extended to measure multiple Bloch coordinates simultaneously. However, since then no achievement has been made.

Chapter 3

Simulation based investigation of the extension possibilities of the Martingale method

This chapter presents the first part of my thesis work, in which in accordance with the aims described in Section 1.2, I investigated the generalization possibilities of the Martingale state estimation technique to multiple dimensions using symbolic and numerical computations in the case, when there is no local time evolution in the measured system. That is, I tried to discover whether the extension of the Martingale method is possible or not besides the assumption that $L_{1,S} = \mathrm{I} = L_{2,S}$, in order to make it capable of measuring multiple coordinates of two-level quantum systems simultaneously.

During the investigation I followed two significantly different paths. In the first case I reduced the problem to root-finding tasks, whereas the second direction aimed at answering the question of extensibility by observing the trajectories of the measured quantum bit coordinates.

3.1 Reducing the problem to root-finding tasks

Based on the equations of Chapter 2 I implemented symbolic and numerical algorithms in Matlab environment [6], which are capable of simulating indirect measurements being parameterizable on a wide range.

Among others the parameterization of the implemented algorithms involves the following:

- initial values for both the measured qubit (x_1, x_2, x_3) and the ancilla qubit (y_1, y_2, y_3),
- characterization of the coupling, i.e. coupling directions (c_1, c_2, c_3) and time of common development (t),
- definition of an arbitrary linear combination of the Pauli matrices, from which the measurement operators are computed.

With the use of these algorithms I was able to determine an equation describing the conditional expected values of the states of the measured quantum bits in terms of the number of indirect measurement iterations.

Denote by $x(n) = (x_1(n), x_2(n), x_3(n))$ the Bloch vector of the state of the measured system at the beginning of iteration n in the measurement sequence. Obviously, a necessary condition for using any type of martingale based method is to extract a martingale from the sequence of states passed through by the measured quantum system during the measurement process. Within the described environment the only way to do this is to properly set the above listed parameters of the measurement setup.

By applying the algorithms I implemented the difference between the conditional expected value of the state in iteration $n+1$ conditioned on the state in iteration n and the state in iteration n for each coordinate $k \in \{1,2,3\}$ in terms of the parameters $\mathbf{p} = (y_1, y_2, y_3, c_1, c_2, c_3, t)$ can be obtained as follows:

$$[\mathbb{E}(x(n+1)|x(n)) - x(n)]_k = f_{k,0}(\mathbf{p}) + f_{k,1}(\mathbf{p})x_1(n) + f_{k,2}(\mathbf{p})x_2(n) + f_{k,3}(\mathbf{p})x_3(n), \quad (3.1)$$

where the coefficient functions $\{f_{k,i}(\mathbf{p})\}_{k=1,i=1}^{3,4}$ are the sums of products involving trigonometric functions of the members of $\mathbf{p}$:

$$f_{1,0}(\mathbf{p}) = -\frac{1}{2}y_1\left(\cos(2c_2t + 2c_3t) - \cos(2c_2t - 2c_3t)\right) \quad (3.2)$$

$$f_{1,1}(\mathbf{p}) = \frac{1}{2}\left(\cos(2c_2t + 2c_3t) + \cos(2c_2t - 2c_3t)\right) - 1 \quad (3.3)$$

$$f_{1,2}(\mathbf{p}) = \frac{1}{2}y_3\left(\sin(2c_2t + 2c_3t) - \sin(2c_2t - 2c_3t)\right) \quad (3.4)$$

$$f_{1,3}(\mathbf{p}) = -\frac{1}{2}y_2\left(\sin(2c_2t + 2c_3t) + \sin(2c_2t - 2c_3t)\right) \quad (3.5)$$

$$f_{2,0}(\mathbf{p}) = -\frac{1}{2}y_2\left(\cos(2c_3t + 2c_1t) - \cos(2c_3t - 2c_1t)\right) \quad (3.6)$$

$$f_{2,1}(\mathbf{p}) = -\frac{1}{2}y_3\left(\sin(2c_3t + 2c_1t) + \sin(2c_3t - 2c_1t)\right) \quad (3.7)$$

$$f_{2,2}(\mathbf{p}) = \frac{1}{2}\left(\cos(2c_3t + 2c_1t) + \cos(2c_3t - 2c_1t)\right) - 1 \quad (3.8)$$

$$f_{2,3}(\mathbf{p}) = \frac{1}{2}y_1\left(\sin(2c_3t + 2c_1t) - \sin(2c_3t - 2c_1t)\right) \quad (3.9)$$

$$f_{3,0}(\mathbf{p}) = -\frac{1}{2}y_3\left(\cos(2c_1t + 2c_2t) - \cos(2c_1t - 2c_2t)\right) \quad (3.10)$$

$$f_{3,1}(\mathbf{p}) = \frac{1}{2}y_2\left(\sin(2c_1t + 2c_2t) - \sin(2c_1t - 2c_2t)\right) \quad (3.11)$$

$$f_{3,2}(\mathbf{p}) = -\frac{1}{2}y_1\left(\sin(2c_1t + 2c_2t) + \sin(2c_1t - 2c_2t)\right) \quad (3.12)$$

$$f_{3,3}(\mathbf{p}) = \frac{1}{2}\left(\cos(2c_1t + 2c_2t) + \cos(2c_1t - 2c_2t)\right) - 1 \quad (3.13)$$

As it can be observed, these functions are completely independent of the measurement operators. They depend only on the initial values of the qubits, the coupling directions and the time of common development.

Since a martingale must be extracted in order to have the possibility to apply the martingale method to the whole quantum bit, i.e. to extend the technique to three dimensions, according to the definition of martingales, a parameterization $\mathbf{p}$ of the measurement system must be found, for which the increment presented in Eq. 3.1 is zero for each coordinate. The expression on the right hand side of Eq. 3.1 is a linear polynomial of $x(n)$, hence it is zero for arbitrary state of the measured qubit if and only if all of its coefficient functions are equal to zero. This is required for each coordinate, i.e. every coefficient function form Eq. 3.2 to Eq. 3.13 must be zero. Thus, the problem of finding a martingale is now reduced to root-finding tasks, which are to be solved simultaneously.

However, this problem seems to be analytically intractable. Therefore, numerical methods were involved in the investigation.

For the purpose of solving the root-finding tasks simultaneously, another trick was applied. It is easy to see, that all of the coefficient functions are zero if and only if the sum of the absolute values of the coefficient functions (denoted by $F(\mathbf{p})$) is also equal to zero:

$$F(\mathbf{p}) := \sum_{k=1}^{3} \sum_{j=1}^{4} |f_{k,j}(\mathbf{p})| = 0. \tag{3.14}$$

Thus, our task is to determine parameterizations $\mathbf{p}$ minimizing $F(\mathbf{p})$.

For giving a sense of the character of function F, it is illustrated by its two-dimensional cross-sections along different coordinates over the parameter space in Figure 3.1 and in Figure 3.2.

By executing numerical optimization algorithms on function F, those parameterizations can be obtained easily, for which the function takes its minimum, i.e. it is equal to zero. However, after setting up the measurement system using these parameter values, a very distressing observation can be made. Namely, in all cases the equation $x(n+1) = x(n)$ always hold true. That is, the measured system is not affected by the measurement at all. In turn, this fact has the consequence that the outcome of the indirect measurement will be completely independent of the quantum bit being estimated, i.e. no information can be extracted from the system.

If the extension possibilities of the martingale technique to only two simultaneously estimated coordinates is analyzed instead of extending the method to three dimensions, unfortunately, the same negative result can be observed.

Therefore, the unfavorable conclusion must be faced: if there is no local time evolution in the measured system, the generalization possibilities of the martingale method became strongly doubtful.

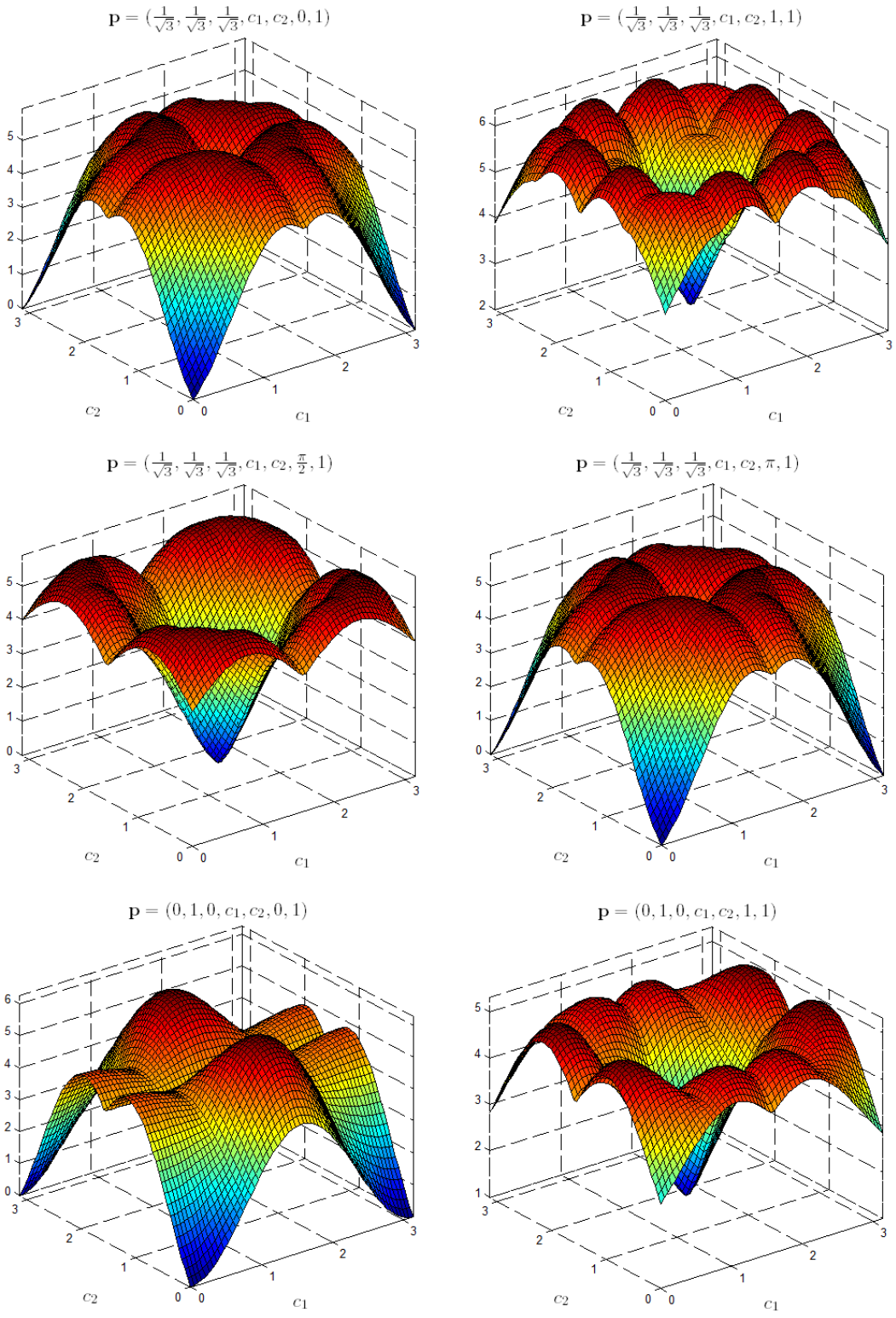

Figure 3.1: *Two-dimensional cross-sections of F along different coordinates over the parameter space.*

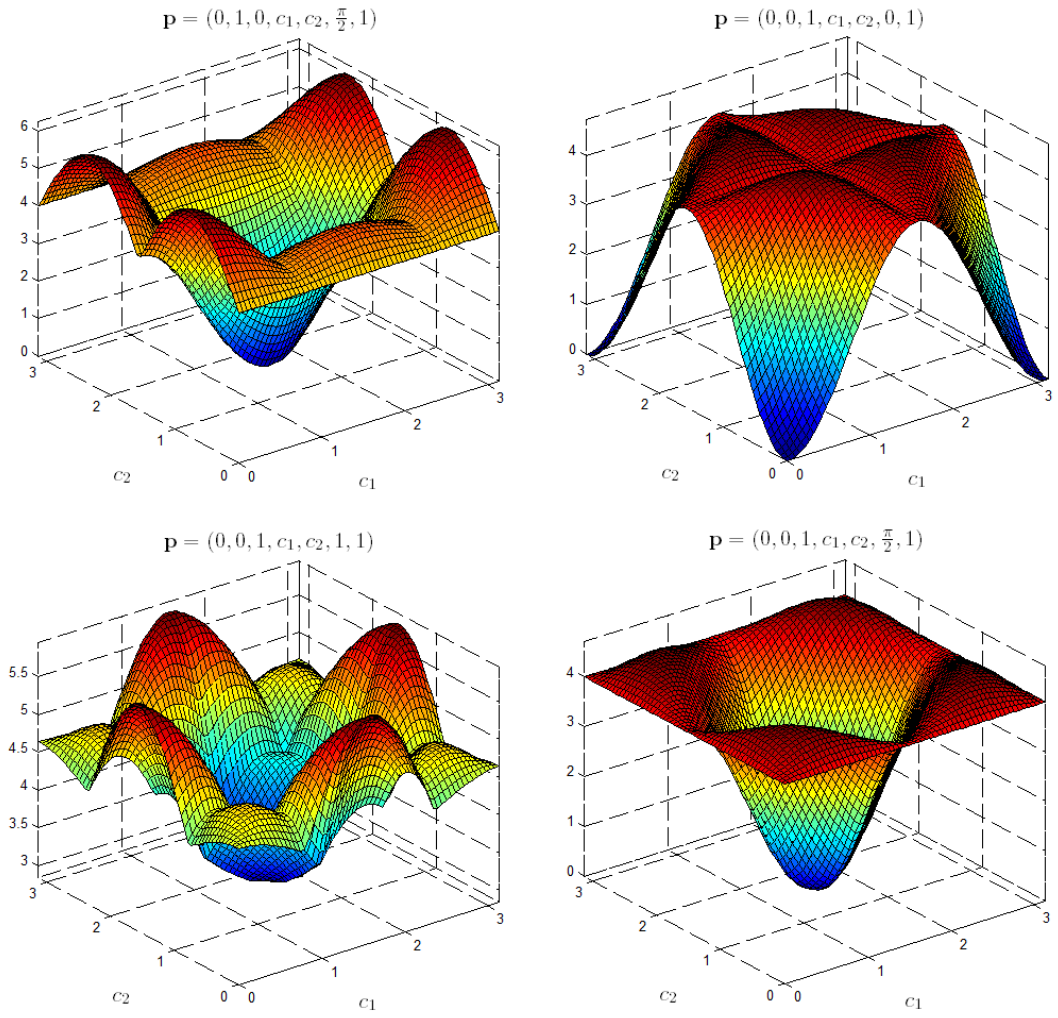

Figure 3.2: *Two-dimensional cross-sections of F along different coordinates over the parameter space (continued).*

3.2 Trajectories of the measured quantum bits during the measurement

In order to confirm the above negative results, I analyzed the trajectories of $x(n)$ in case of different parameterizations and initial values.

The numerical simulations showed that, except some special cases, the trajectories of $x(n)$ converge to particular attractor states. Each measurement outcome sequence has its own attractor state unambiguously determined by the ancilla qubit and by the characterization of the coupling regardless of $x(0)$, i.e. independently of the initial state of the examined quantum bit. Figure 3.3 illustrates this observation, where 10000 different initial points were selected randomly from the Bloch ball based on uniform distribution, whereas the ancilla qubit values and the couplings were kept fixed. In all cases when the same sequences of outcomes were assumed, the sequences of the states of the measured qubits converged to identical attractors regardless of the initial states. In Figure 3.3 the squares on the left (blue) and on the right (red) show the locations of the attractors corresponding to all negative and all positive outcome sequences, respectively. The square in the middle (green) denotes the location of the attractor of the conditional expected values of the states conditioned on the preceding states, i.e. the attractor of the process $\mathbb{E}(x(n+1)|x(n))$.

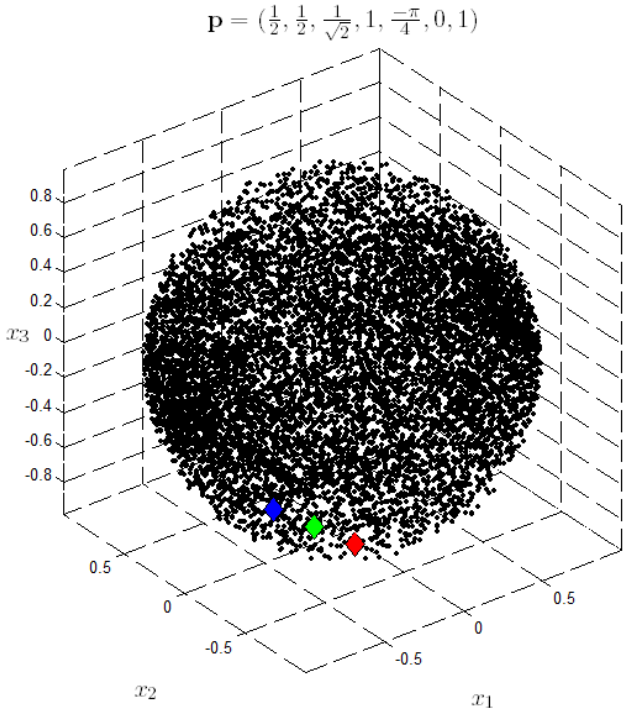

Figure 3.3: *Attractors of measurement sequences with 10000 different randomly selected initial points from the Bloch ball.*

Each sub-figure of Figure 3.4 presents the trajectories during measurement sequences

started from a single random initial state. (The alignments of the attractors are the same as in Figure 3.3.)

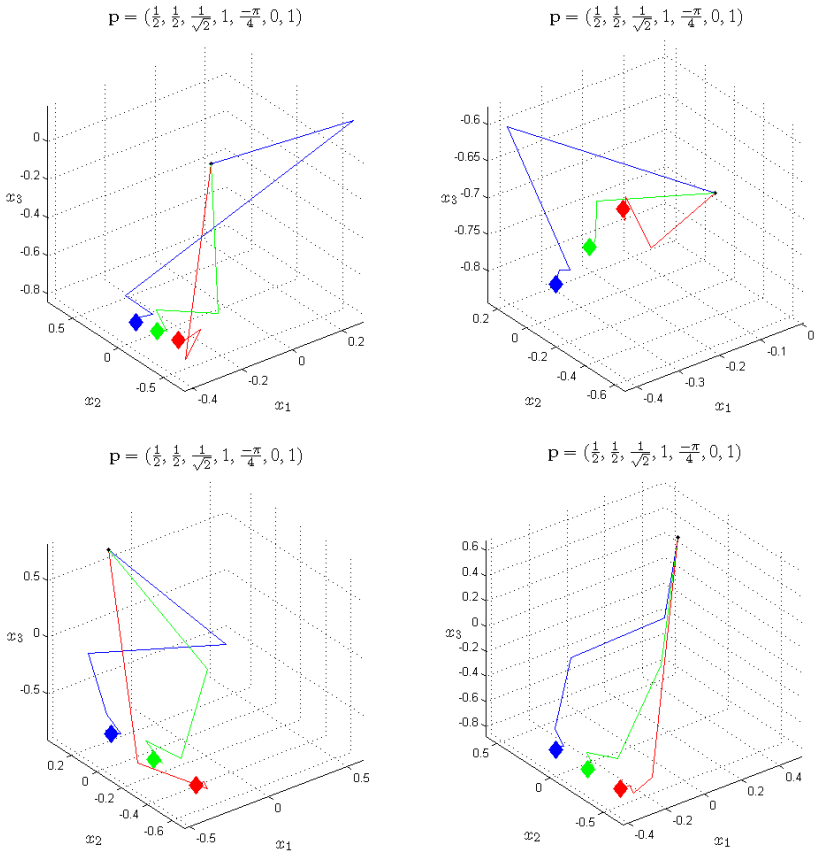

Figure 3.4: *Trajectories corresponding to measurement sequences started from single random initial states.*

Every time in the exceptional special cases mentioned above, i.e. when the trajectories do not converge to the identical attractor states, the value of exactly one coordinate does not alter (see Figure 3.5). These are the parameterizations, for which the original one-dimensional martingale method can be applied.

For the multi-dimensional generalization such parameterizations would be necessary, for which the other coordinates in the trajectories of the conditional expected values of the states conditioned on the preceding states also preserve their values. However, no such parameterization has been discovered.

This fact confirms the negative results of the previous section.

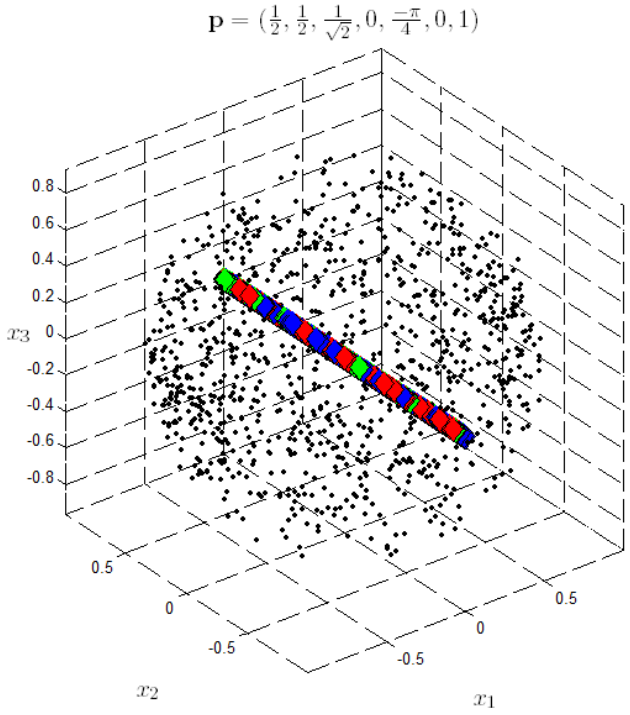

Figure 3.5: *Attractors in an exceptional case.*

Chapter 4

Analytical characterization of indirect measurement cycles of two-level quantum systems

In the previous chapter parameterizations were searched for, for which the sequence of the states of the measured system forms a martingale through the sequence of indirect measurements. Although, such parameterizations have been found, it seemed that in these cases the probabilities of measurement outcomes were independent of the measured system, i.e. no state information could be extracted from it. Thus, the attempts to extend the Martingale method have been failed so far. Moreover, the signs pointed to the impossibility of the extension.

These results motivated a deeper understanding of indirect measurement cycles of two-level quantum systems via formal analysis. Hence the aim of this chapter is to characterize these measurement cycles analytically while focusing on the expected value of the resulting state of the system and to discuss the extension possibilities of the Martingale state estimation technique on the basis of this analysis.

4.1 General case

In what follows $\rho_S(n), \rho_A(n)$ and $\rho_T(n)$ will denote the density operators of the measured, the ancilla and the composite systems at the beginning of iteration n, respectively, whereas $\rho_T(n)'$ will be the composite state after the time evolution. Similarly, other variables with subscripts S, A and T will also correspond to the respective systems. The Bloch vector of the measured system at the beginning of iteration n will be denoted by $x(n)$ like in the previous chapters.

As it was described in Subsection 2.3.1 an indirect measurement cycle contains three phases: coupling, common time evolution and measurement of the ancilla system together with decoupling.

The following proposition shows that the ordering between the measurement of the ancilla system and the decoupling is indifferent in the analysis of the expected values of the resulting states of the measured system.

Proposition 1. *In case of arbitrary measurement operators, after an indirect measurement cycle the state of the measured system remains the same, expectedly, as its decoupled state after the common time evolution during the measurement cycle, i.e.*

$$\mathbb{E}(\rho_S(n+1)|\rho_S(n)) = \mathrm{Tr}_A(\mathbb{E}(\rho_T(n+1)|\rho_T(n))) = \mathrm{Tr}_A(\rho_T(n)'). \tag{4.1}$$

Proof. Based on Postulate 2, Postulate 4 and Eq. 2.28 in case of coupling parameters c_1, c_2, c_3 and time evolution length t the composite state after time evolution in an indirect measurement cycle is

$$\rho_T(n)' = \sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)}(L_{1,S}\sigma_i L_{2,S}\rho_S(n)L_{2,S}^*\sigma_j L_{1,S}^* \otimes L_{1,P}\sigma_i L_{2,A}\rho_A(n)L_{2,A}^*\sigma_j L_{1,A}^*). \tag{4.2}$$

With substitutions

$$U_{i,S} := L_{1,S}\sigma_i L_{2,S}, \tag{4.3}$$

$$U_{i,A} := L_{1,A}\sigma_i L_{2,A}, \tag{4.4}$$

where the operators $U_{i,S}$ and $U_{i,A}$ are obviously unitary, the compound state after time evolution reads as follows:

$$\rho_T(n)' = \sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)}(U_{i,S}\rho_S(n)U_{j,S}^* \otimes U_{i,A}\rho_P(n)U_{j,A}^*). \tag{4.5}$$

Then, since a measurement is applied on the ancilla system, based on Postulate 3 in case of measurement outcome m the state becomes

$$\frac{M_{m,T}\rho_T(n)'M_{m,T}^*}{\mathrm{Tr}(M_{m,T}\rho_T(n)'M_{m,T}^*)} = \frac{(\mathrm{I}\otimes M_{m,A})\rho_T(n)'(\mathrm{I}\otimes M_{m,A})}{\mathrm{Tr}((\mathrm{I}\otimes M_{m,A})\rho_T(n)'(\mathrm{I}\otimes M_{m,A}^*)} =$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} \frac{\mu_i(c,t)\overline{\mu_j(c,t)}}{\mathrm{Tr}((\mathrm{I}\otimes M_{m,A})\rho_T(n)'(\mathrm{I}\otimes M_{m,A}^*))}(U_{i,S}\rho_S(n)U_{j,S}^* \otimes M_{m,A}U_{i,A}\rho_A(n)U_{j,A}^*M_{m,A}^*). \tag{4.6}$$

Due to the bilinearity of the tensor product the conditional expected value of the new compound state is

$$\mathbb{E}(\rho_T(n+1)|\rho_T(n)) = \sum_m \mathrm{Tr}(M_{m,T}\rho_T(n)'M_{m,T}^*)\frac{M_{m,T}\rho_T(n)'M_{m,T}^*}{\mathrm{Tr}(M_{m,T}\rho_T(n)'M_{m,T}^*)} =$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)}\left(U_{i,S}\rho_S(n)U_{j,S}^* \otimes \sum_m M_{m,A}U_{i,A}\rho_A(n)U_{j,A}^*M_{m,A}^*\right). \tag{4.7}$$

Finally, the conditional expected value of the state of the measured system after one indirect measurement cycle is obtained by applying the partial trace operator (and using its linearity as well as Eq. 2.20):

$$\mathrm{Tr}_A(\mathbb{E}(\rho_T(n)'|\rho_T(n))) =$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)} \left(U_{i,S}\rho_S(n)U_{j,S}^* \otimes \sum_m \mathrm{Tr}(M_{m,A}U_{i,A}\rho_A(n)U_{j,A}^*M_{m,A}^*) \right), \quad (4.8)$$

where using the properties of the trace operator and considering that $\sum_m M_{m,A}^*M_{m,A} = \mathrm{I}$ holds:

$$\sum_m \mathrm{Tr}(M_{m,A}U_{i,A}\rho_A(n)U_{j,A}^*M_{m,A}^*) =$$

$$= \mathrm{Tr}\left(\left(\sum_m M_{m,A}^*M_{m,A} \right) U_{i,A}\rho_A(n)U_{j,A}^* \right) = \mathrm{Tr}(U_{i,A}\rho_A(n)U_{j,A}^*), \quad (4.9)$$

and therefore:

$$\mathbb{E}(\rho_S(n+1)|\rho_S(n)) = \mathrm{Tr}_A(\mathbb{E}(\rho_T(n+1)|\rho_T(n))) =$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)}U_{i,S}\rho_S(n)U_{j,S}^* \,\mathrm{Tr}(U_{i,A}\rho_A(n)U_{j,A}^*) = \mathrm{Tr}_A(\rho_T(n)'). \quad (4.10)$$

□

This result agrees with the observations of Section 3.1, namely, the coefficient functions in Eq. 3.2 – Eq. 3.13 were completely independent of the measurement operators.

On the basis of the previous proposition the Bloch coordinates of the conditional expected value can be expressed as follows.

Lemma 1. *Bloch coordinate $k \in \{1,2,3\}$ of the conditional expected value of the state of the measured system after an indirect measurement cycle is*

$$\mathbb{E}(\rho_S(n+1)|\rho_S(n))_k = \sum_{l=0}^{3} x_l(n)\varphi_{l,k}, \quad (4.11)$$

where

$$\varphi_{l,k} := \frac{1}{2}\sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)}\,\mathrm{Tr}(U_{i,A}\rho_A(n)U_{j,A}^*)\,\mathrm{Tr}\left(L_{1,S}\sigma_i L_{2,S}\sigma_l L_{2,S}^*\sigma_j L_{1,S}^*\sigma_k\right). \quad (4.12)$$

Proof. Using Eq. 4.10 and Eq. 2.7, moreover expressing $\rho_S(n)$ with Bloch coordinates, one obtains for Bloch coordinate $k \in \{1,2,3\}$ of the conditional expected value that

$$\mathbb{E}(\rho_S(n+1)|\rho_S(n))_k = \mathrm{Tr}\left(\sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(t)\overline{\mu_j(c,t)}\,\mathrm{Tr}(U_{i,A}\rho_A(n)U_{j,A}^*)U_{i,S}\rho_S(n)U_{j,S}^*\sigma_k \right) =$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(t)\overline{\mu_j(t)}\,\mathrm{Tr}(U_{i,A}\rho_A(n)U_{j,A}^*)\,\mathrm{Tr}\left(U_{i,S}\frac{1}{2}\left(\mathrm{I} + \sum_{l=1}^{3} x_l(n)\sigma_l\right)U_{j,S}^*\sigma_k \right). \quad (4.13)$$

With $x_0(n) :\equiv 1$ and $\sigma_0 = \mathrm{I}$ this reads as

$$\mathbb{E}(\rho_S(n+1)|\rho_S(n))_k =$$

$$= \sum_{l=0}^{3} x_l(n) \frac{1}{2} \sum_{i=0}^{3} \sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)} \operatorname{Tr}(U_{i,A}\rho_A(n)U_{j,A}^*) \operatorname{Tr}(L_{1,S}\sigma_i L_{2,S}\sigma_l L_{2,S}^*\sigma_j L_{1,S}^*\sigma_k). \tag{4.14}$$

□

The following lemma highlights the dependence between the probability of measurement outcome m and the state of the measured system.

Lemma 2. *During indirect measurement iteration n the probability of measurement outcome m is*

$$\begin{aligned}\operatorname{Tr}(M_{m,T}\rho_T(n)'M_{m,T}^*) = \sum_{i=0}^{3} |\mu_i(c,t)|^2 tr_{i,i} + \\ +2\mathfrak{Re}\{\mu_0(c,t)\overline{\mu_1(c,t)}x_{1,L_2}(n)tr_{0,1} - i\mu_2(c,t)\overline{\mu_3(c,t)}x_{1,L_2}(n)tr_{2,3} + \\ +\mu_0(c,t)\overline{\mu_2(c,t)}x_{2,L_2}(n)tr_{0,2} + i\mu_1(c,t)\overline{\mu_3(c,t)}x_{2,L_2}(n)tr_{0,1} + \\ +\mu_0(c,t)\overline{\mu_3(c,t)}x_{3,L_2}(n)tr_{0,3} - i\mu_1(c,t)\overline{\mu_2(c,t)}x_{3,L_2}(n)tr_{2,3}\},\end{aligned} \tag{4.15}$$

where

$$tr_{i,j} := \operatorname{Tr}(M_{m,A}U_{i,A}\rho_A(n)U_{j,A}^*M_{m,A}^*), \tag{4.16}$$

$\mathfrak{Re}\{a\}$ denotes the real part of the complex number $a \in \mathbb{C}$ and the variable $x_{k,L_2}(n)$ ($k \in \{1,2,3\}$) is the Bloch coordinate k of the state $L_{2,S}\rho_S(n)L_{2,S}^*$.

Proof. Considering the linearity of the trace operator and applying the identity $\operatorname{Tr}(A \otimes B) = \operatorname{Tr}(A) \cdot \operatorname{Tr}(B)$, the probability of measurement outcome m is

$$\operatorname{Tr}(M_{m,T}\rho_T(n)'M_{m,T}^*) =$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)} \operatorname{Tr}(U_{i,S}\rho_S(n)U_{j,S}^* \otimes M_{m,A}U_{i,A}\rho_A(n)U_{j,A}^*M_{m,A}^*) =$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)} \operatorname{Tr}(U_{i,S}\rho_S(n)U_{j,S}^*) \operatorname{Tr}(M_{m,A}U_{i,A}\rho_A(n)U_{j,A}^*M_{m,A}^*). \tag{4.17}$$

Since $L_{1,S}$ is unitary, based on Eq. 4.3:

$$\operatorname{Tr}(U_{i,S}\rho_S(n)U_{j,S}^*) = \operatorname{Tr}(L_{1,S}^*L_{1,S}\sigma_i L_{2,S}\rho_S(n)L_{2,S}^*\sigma_j) = \operatorname{Tr}(L_{2,S}\rho_S(n)L_{2,S}^*\sigma_j\sigma_i). \tag{4.18}$$

With the substitution of Eq. 4.16, applying Table 2.1 and Eq. 2.7, furthermore using that for complex numbers $a_1, a_2 \in \mathbb{C} : a_1\overline{a_2} + \overline{a_1}a_2 = 2\mathfrak{Re}\{a_1\overline{a_2}\}$, Eq. 4.15 follows. □

4.2 Special case: without local time evolution in the measured system

4.2.1 Determination of the variables $\varphi_{l,l}$ $(l \in \{0,1,2,3\})$

If there is *no local time evolution in the measured system*, i.e. $L_{1,S} = \mathrm{I} = L_{2,S}$, the values of $\varphi_{l,k}(l,k \in \{0,1,2,3\})$ are

$$\varphi_{l,k} = \frac{1}{2}\sum_{i=0}^{3}\sum_{j=0}^{3} \mu_i(c,t)\overline{\mu_j(c,t)}\,\mathrm{Tr}(U_{i,A}\rho_A(n)U_{j,A}^*)\,\mathrm{Tr}(\sigma_i\sigma_l\sigma_j\sigma_k) \tag{4.19}$$

Then, for the values of $\varphi_{l,l}(l \in \{0,1,2,3\})$ the following lemma holds.

Lemma 3. *The values of* $\varphi_{l,l}(l \in \{0,1,2,3\})$ *are*

$$\varphi_{0,0} = |\mu_0(c,t)|^2 + |\mu_1(c,t)|^2 + |\mu_2(c,t)|^2 + |\mu_3(c,t)|^2, \tag{4.20}$$

$$\varphi_{1,1} = |\mu_0(c,t)|^2 + |\mu_1(c,t)|^2 - |\mu_2(c,t)|^2 - |\mu_3(c,t)|^2, \tag{4.21}$$

$$\varphi_{2,2} = |\mu_0(c,t)|^2 - |\mu_1(c,t)|^2 + |\mu_2(c,t)|^2 - |\mu_3(c,t)|^2, \tag{4.22}$$

$$\varphi_{3,3} = |\mu_0(c,t)|^2 - |\mu_1(c,t)|^2 - |\mu_2(c,t)|^2 + |\mu_3(c,t)|^2. \tag{4.23}$$

Proof. From Eq. 4.19 for all $l \in \{0,1,2,3\}$

$$\varphi_{l,l} = \frac{1}{2}\sum_{i=0}^{3} |\mu_i(c,t)|^2\,\mathrm{Tr}(U_{i,A}\rho_A(n)U_{i,A}^*)\,\mathrm{Tr}(\sigma_i\sigma_l\sigma_j\sigma_l), \tag{4.24}$$

where

$$\mathrm{Tr}(U_{i,A}\rho_A(n)U_{i,A}^*) = \mathrm{Tr}(U_{i,A}^*U_{i,A}\rho_A(n)) = \mathrm{Tr}(\mathrm{I}\,\rho_A(n)) = \mathrm{Tr}(\rho_A(n)) = 1, \tag{4.25}$$

because density operators are trace-one matrices.

Since the Pauli matrices are closed under matrix multiplication up to factors ± 1 and $\pm i$, the product $\sigma_i\sigma_l\sigma_j\sigma_l = (\sigma_i\sigma_l)(\sigma_j\sigma_l)$ is not traceless if and only if it is equal to $\pm\mathrm{I}$ or $\pm i\,\mathrm{I}$. This can occur if and only if $\sigma_i\sigma_l = \pm\sigma_j\sigma_l$ or $\sigma_i\sigma_l = \pm i\sigma_j\sigma_l$ (see Table 2.1). By multiplying with σ_l from the right, $\sigma_i = \pm\sigma_j$ and $\sigma_i = \pm i\sigma_j$ can be obtained, from which $\sigma_i = \sigma_j$ and thus $i = j$ follows. Then, for the trace of the product:

$$\mathrm{Tr}(\sigma_i\sigma_l\sigma_j\sigma_l) = \begin{cases} \mathrm{Tr}((\sigma_i\sigma_l)^2) = \pm 2, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{4.26}$$

Applying these considerations:

$$\varphi_{l,l} = \sum_{i=0}^{3} \pm\mu_i(c,t)\overline{\mu_i(c,t)}, \tag{4.27}$$

where the appropriate signs can be read out from Table 2.1 resulting in the expressions formulated in Eq. 4.20 – Eq. 4.23. □

Notice that since 'Bloch coordinate 0' is always 1, i.e $x_0(n) \equiv 1$ as well as $\mathbb{E}(\rho_S(n+1)|\rho_S(n))_k \equiv 1$, whereas the other ('real') Bloch coordinates can take arbitrary values from the interval $[-1, 1]$, considering Eq. 4.11 it follows that

$$\varphi_{l,0} = \begin{cases} 1, & \text{if } l = 0 \\ 0, & \text{if } l \neq 0 \end{cases} \tag{4.28}$$

(By the way, $\varphi_{0,0}$ also follows from Eq. 2.24 – Eq. 2.27 using the trigonometric identity $\sin^2 \alpha + \cos^2 \alpha \equiv 1$ multiple times.)

Although, from Eq. 2.24 – Eq. 2.27, Table 2.1, Eq. 4.19 and using trigonometric identities it can also be shown that $\{\varphi_{l,k}\}_{l=1,k=1}^{3,4} = \{f_{k,i}(\mathbf{p})\}_{k=1,i=1}^{3,4}$ holds true (see Eq. 3.2 – Eq. 3.13 in Section 3.1), since this fact is irrelevant from the point of view of the aim of the present analysis, this equivalence will not be justified.

4.2.2 Martingale case

Based on the results described above the following theorem shows that without local time evolution no extension of the Martingale method may exist estimating a whole two-level quantum system, i.e all of its coordinates simultaneously.

Theorem 1. *If the sequence of states of the measured system $\{\rho_S(n)\}$ form a martingale and there is no local time evolution in the measured system, then the probability of every measurement outcome is independent of the state of the measured system, consequently, the state of the system cannot be estimated by indirect measurements.*

Proof. Assume that the sequence of states of the measured system form a martingale through the measurement sequence. Then, $\forall k \in \{0, 1, 2, 3\} : \mathbb{E}(\rho_S(n+1)|\rho_S(n))_k = x_k(n)$ must hold, and thus from Eq. 4.11 of Lemma 1 it follows that $\varphi_{l,k}$ is the *Kronecker-delta*, that is

$$\varphi_{l,k} = \delta_{l,k} = \begin{cases} 1, & \text{if } l = k \\ 0, & \text{if } l \neq k \end{cases} \tag{4.29}$$

Applying Eq. 4.20 – Eq. 4.23 of Lemma 3 this implies the following system of equations:

$$1 = |\mu_0(c,t)|^2 + |\mu_1(c,t)|^2 + |\mu_2(c,t)|^2 + |\mu_3(c,t)|^2 \tag{4.30}$$

$$1 = |\mu_0(c,t)|^2 + |\mu_1(c,t)|^2 - |\mu_2(c,t)|^2 - |\mu_3(c,t)|^2 \tag{4.31}$$

$$1 = |\mu_0(c,t)|^2 - |\mu_1(c,t)|^2 + |\mu_2(c,t)|^2 - |\mu_3(c,t)|^2 \tag{4.32}$$

$$1 = |\mu_0(c,t)|^2 - |\mu_1(c,t)|^2 - |\mu_2(c,t)|^2 + |\mu_3(c,t)|^2 \tag{4.33}$$

The terms $|\mu_i(c,t)|^2$ $(i \in \{0, 1, 2, 3\})$ are obviously nonnegative.

After adding Eq. 4.31, Eq. 4.32 and Eq. 4.33 to each other pairwise, dividing the sums by 2 and rearranging the equations, one can obtain that

$$|\mu_0(c,t)|^2 = 1 + |\mu_1(c,t)|^2 = 1 + |\mu_2(c,t)|^2 = 1 + |\mu_3(c,t)|^2. \tag{4.34}$$

However, from Eq. 4.30 inequality $|\mu_0(c,t)|^2 \leq 1$ follows, hence

$$|\mu_0(c,t)|^2 = 1, \quad |\mu_1(c,t)|^2 = 0, \quad |\mu_2(c,t)|^2 = 0, \quad |\mu_3(c,t)|^2 = 0, \tag{4.35}$$

implying that

$$\mu_1(c,t) = 0, \quad \mu_2(c,t) = 0, \quad \mu_3(c,t) = 0. \tag{4.36}$$

Confronting this with Eq. 4.15 of Lemma 2, the probability of arbitrary outcome m becomes

$$\mathrm{Tr}(M_{m,T}\rho_T(n)'M_{m,T}^*) = \sum_{i=0}^{3} |\mu_i(c,t)|^2 tr_{i,i}, \tag{4.37}$$

which is completely independent of the measured system, agreeing with the results of Section 3.1. □

Moreover, let us see the case, when only two coordinates are tried to be estimated simultaneously.

Theorem 2. *If within the sequence of states of the measured system $\{\rho_S(n)\}$ more than one coordinates form a martingale and there is no local time evolution in the measured system, then the probability of every measurement outcome is independent of the state of the measured system, consequently, the state of the system cannot be estimated by indirect measurements.*

Proof. Assume that within the sequence of states of the measured system more than one coordinates form a martingale through the measurement sequence. Then, in the proof of Theorem 1 besides Eq. 4.30 (which is always true) at least two additional equations hold out of Eq. 4.31, Eq. 4.32 and Eq. 4.33.

After adding these two equations to each other, dividing the sum by 2 and rearranging the equation, one can obtain that

$$|\mu_0(c,t)|^2 = 1 + |\mu_i(c,t)|^2 \text{ for at least one } i \in \{1,2,3\}. \tag{4.38}$$

However, as $|\mu_0(c,t)|^2 \leq 1$ holds, $|\mu_0(c,t)|^2$ must be equal to 1. Consequently, due to Eq. 4.30, Eq. 4.35 and thus Eq. 4.36 as well as Eq. 4.37 hold true again, implying that without local time evolution no extension of the Martingale method may exist estimating even two coordinates of a two-level quantum system. □

Therefore, the conjecture formulated in Chapter 3 is true, that is, if there is no local time evolution in the measured system, the Martingale method cannot be extended to multiple dimensions.

Chapter 5

Conclusions

My thesis aimed to extend the Martingale quantum state estimation method (see Chapter 2) to multiple dimensions, i.e. to make it capable of estimating multiple coordinates of two level quantum systems simultaneously.

In the first half of my thesis work (see Chapter 3), as the first step of this task I searched for appropriate parameterizations for the indirect measurement system, for which the sequence of the states of the measured system forms a martingale through the sequence of indirect measurements. Since there are nonlinear terms in the dynamics of coupled quantum systems, for this purpose I implemented a simulation environment on a computer (in Matlab) using symbolic and numerical computations. Then I reduced the problem to root finding tasks and solved it in the established environment.

Besides the resulting parameterizations, however, no information can be extracted from the measured system. I made similar observations also in case of two coordinates instead of three. These led to the conjecture that the extension of the Martingale technique is not possible to multiple dimensions.

I reinforced this conjecture by another series of simulations, where I monitored the trajectories of the states of the measured systems and I noticed that the conditional expected values of the states could not be fixed in general cases.

Due to these negative results, in the second half of my work (see Chapter 4) I characterized the measurement cycles of indirectly measured two-level quantum systems deep enough to discover crucial algebraic relationships, based on which I was able to prove the conjecture that the Martingale quantum state estimation method is inextensible for multiple dimensions (if there is no time evolution in the measured system), indeed.

My thesis work can be summarized in a nutshell in the following sentence:
"My task was to extend the Martingale state estimation method to multiple dimensions in the case, when there is no local time evolution in the measured system, which task I conjectured to be impossible, then I proved this conjecture to be true."

The new research results I achieved may be beneficial in the intensively studied field of quantum state estimation for researchers interested in developing and improving state estimation techniques.

Although, my results are complete and stand-alone, obviously there are a number of

possibilities to continue my research.

A natural possibility is to investigate the case when during the measurement process local time evolution is allowed within the measured system.

Moreover, an obvious future aim may be the analysis of other quantum state estimation techniques from the point of view of extensibility to multiple dimensions.

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