Stabilizing dynamic feedback design of quasi-polynomial systems using their underlying reduced linear dynamics

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***Abstract*— Based on the underlying dynamically similar linear system of a quasi-polynomial (QP) system [7], a dynamic feedback controller for single input QP systems is proposed in this work that can locally stabilize the closed-loop system using a pre-defined quadratic control Lyapunov function. Since the parameter matrix of the dynamically similar reduced linear dynamics depends linearly on the feedback gain parameters, the controller can be designed by solving LMIs. Conditions for extending the controller design for obtaining a globally stable closed-loop system are also investigated.**

I. INTRODUCTION

Quasi-polynomial (QP) systems form a wide class of smooth nonnegative systems and they clearly play an increasingly important role in the modeling of dynamical processes, particularly that of biochemical origin. The QP system class was introduced and first analyzed in [1], [2]. In [1] it was shown that majority of smooth ODE models can be algorithmically embedded into the QP form, and the so-called quasi-monomial (QM) transformation was defined under which the QP-model form is invariant. Furthermore, the QM transformation splits the family of QP systems into equivalence-classes, and in each class two simple canonical forms were defined in [2]. QP systems are also called Generalized Lotka-Volterra (GLV) systems, because the monomials of a QP system form a classical Lotka-Volterra (LV) system in a transformed state space which is often of higher dimension than that of the original QP system [4], [3]. Thus, numerous properties of QP models like integrability, stability, persistence or the existence of invariants can be examined using the corresponding LV system, the qualitative properties of which have been intensively studied for a long time [13]. Based on the above, we can say that LV models "have the status of a canonical format" within smooth nonlinear dynamical systems [9]. Moreover, the simple matrix structure characterizing QP models allows us to perform important model analysis tasks using efficient numerical algorithms. In [6], the QP formalism was first extended to the discrete-time case demonstrating that the LV system representation plays an important role in that case, too. The conditions for transforming neural network models to QP form are considered in [10] where the most important conclusion is that generalized LV systems are universal approximators of certain dynamical systems, similarly to e.g. continuous-time neural networks. It was shown using the examples of biological networks in [11], that the formerly known QP stability criterion can be extended to examine robust stability with respect to the system parameters.

It was shown in [7], that an appropriately computed reduced linear dynamics can support the qualitative dynamical analysis of such QP systems that otherwise do not show complex nonlinear behavior (like limit cycles, chaos etc.). A possible general form of nonlinear control systems in QP form was first given in [15]. This representation was used in [16] for the design of globally stabilizing nonlinear static feedback leading to the solution of bilinear matrix inequalities (BMIs). The purpose of this paper is to combine and extend these results by addressing the efficient computation of a stabilizing dynamical feedback for QP systems with a simple general structure.

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II. BASIC NOTIONS

A. Autonomous quasi-polynomial systems

The system dynamics of an autonomous QP system can be described by a set of differential-algebraic equations (DAEs), where the ordinary differential equations

$$\frac{dx_i}{dt} = x_i \left(\lambda_i + \sum_{j=1}^{m} \alpha_{ij} q_j \right) \ , \ i = 1, ..., n \qquad (1)$$

are equipped by the so called quasi-monomial (QM) relationships

$$q_j = \prod_{i=1}^{n} x_i^{\beta_{ji}} \qquad (2)$$

that are apparently nonlinear (monomial-type) algebraic equations. Two sets of variables are defined, that are (i) the differential variables $x_i, i = 1, ..., n$, and (ii) the quasi-monomials (QMs) $q_j, j = 1, ..., m$. The parameters of the above model are collected in two rectangular matrices $[A]_{ij} = \alpha_{ij}$, $[B]_{ji} = \beta_{ji}$ and a vector $[\Lambda]_i = \lambda_i$.

In order to avoid degenerate cases, we assume $m \geq n$.

It can be shown [4], that systems in the form (1) have a special property which provides that the products $M = B\,A$ and $N = B\,\Lambda$ are invariant for groups of models. This way the class of quasi-polynomial models can be partitioned according to the values of these products. Each of these partitions can be represented by a so called Lotka-Volterra model which is a special QP system with coefficient matrices $B = I$, $A = M$ and $\Lambda = N$ ([4]).

1) Decomposition of the parameter matrices: Furthermore, the rectangular matrices A and B are assumed to have full rank. Therefore, they admit a decomposition in the form

$$B = \left[\frac{B^*}{\overline{B}} \right] = \left[\frac{B^*}{N_B\, B^*} \right], \tag{3}$$

$$A = \left[\begin{array}{c|c} A^* & \overline{A} \end{array} \right] \tag{4}$$

where A^* and B^* are square invertible matrices. Note, that $\overline{B} = N_B\, B^*$ and $N_B = \overline{B}\, B^{*-1}$.

2) Stability analysis: Henceforth it is assumed that x^* is a positive equilibrium point, i.e. $x^* \in \text{int}(\mathbb{R}^n_+)$ in the quasi-polynomial case and similarly $q^* \in \text{int}(\mathbb{R}^m_+)$ is a positive equilibrium point in the quasi-monomials. For QP systems there is a well known candidate Lyapunov function family ([3],[5]), which is in the form:

$$V(q) = \sum_{i=1}^{m} c_i \left(q_i - q_i^* - q_i^* \ln \frac{q_i}{q_i^*} \right), \tag{5}$$
$$c_i > 0, \quad i = 1 \ldots m,$$

where $q^* = \left[\begin{array}{ccc} q_1^* & \ldots & q_m^* \end{array} \right]^T$ is the equilibrium point corresponding to the equilibrium x^* of the original quasi-polynomial system (1).

One can determine the global asymptotic stability of an autonomous QP system with its characterizing similarity matrix $M = B\, A$ by checking the feasibility of the following LMI

$$C\, M + M^T\, C \prec 0 \tag{6}$$

for a diagonal matrix C with positive elements ([14]), i.e. $C = \text{diag}\{c_1, ..., c_m\}$, $c_i > 0$, where '$\prec 0$' denotes the negative definiteness of a symmetric matrix.

The chance of proving global stability can be somewhat extended using the time reparametrization transformation (see [18]), however, the numerical problem to be solved turns to a bilinear matrix inequality.

B. Dynamically similar reduced linear system

Translated X-factorable transformation [12] of an autonomous QP system, together with the reduction of the linearly dependent part of a non-minimal linear ODE can be applied to construct a dynamically similar reduced (and minimal) linear ordinary differential equations (ODE) [7] as follows. The decomposition of the parameter matrices A and B in (4) and (3) is used to compute the coefficient matrix

$$M^*_{red} = B^* A^* + B^* \overline{A} \overline{B} B^{*-1} = B^* (A^* + \overline{A} \overline{B} B^{*-1}) \tag{7}$$

where $M^*_{red} \in \mathbb{R}^{n \times n}$ characterizes the reduced dynamically similar ODE. The phase portrait of this ODE is dynamically similar to that of the original autonomous QP systems in the positive orthant if they have a joint equilibrium point there.

It is important to emphasize, that the above translated X-factorable transformation transforms the phase portrait of the state space, but not the state variables. Therefore, *the original autonomous QP system and its reduced linear counterpart are only locally dynamically similar in the neighborhood of an equilibrium point but not equivalent.*

III. QP SYSTEMS WITH DYNAMIC QP FEEDBACK

Motivated by the above described advantageous properties of autonomous QP systems, we aim at finding a controller structure that renders an open-loop QP-system to be in autonomous QP-form also in closed-loop. This can be achieved by using either a static QP feedback (see [16]), or by defining a suitable dynamic QP feedback controller.

A. Extended and feedback quasi-polynomial systems

As we have seen before, the original QP system model (1) corresponds to an autonomous system, that should be extended by a suitable input structure to describe the effect of the manipulable inputs . In [16] an input-affine extension is proposed where the nonlinear input function is a quasi-polynomial expression. However, a much simpler input structure can be used in most of the real applications.

In the area of process systems, for example, the typical control inputs are the flow rates of the inlet and outlet streams. The state equations originate from mass, component mass and energy balances in this case, and the input appears in these equations as a bi-linear $x_i u$-type term (see [17] for details). This situation has motivated us to introduce the simple feedback structure below.

1) The feedback structure: In order to obtain a sufficiently general but computationally tractable control system , the following single input QP feedback structure is proposed.

1) Only one new monomial $q_{m+1} = u$ is introduced that is the input variable itself.
2) The new monomial appears linearly in the original state equations

$$\dot{x}_i = x_i \left(\lambda_i + \sum_{j=1}^{m} \alpha_{ij} q_j + \gamma_i u \right), \quad i = 1, \ldots, n. \tag{8}$$

3) The dynamical quasi-polynomial feedback is constructed with the same monomials as in the original system extended by the new one ($q_{n+1} = u$)

$$\dot{u} = u \left(\kappa + \sum_{j=1}^{m} k_j q_j + k_\gamma u \right) \tag{9}$$

Then the closed-loop system can be seen as an autonomous QP system, with an additional state variable $x_{n+1} = u$ and an additional quasi-monomial $q_{m+1} = u$.

2) Equilibrium points: Let us denote the positive equilibrium point of the original autonomous QP system by x^*, that satisfies the nonlinear algebraic equation

$$-\lambda_i = \sum_{j=1}^{m} \alpha_{ij} q_j^*, \quad i = 1, \ldots, n \tag{10}$$

where $q_j^* = \prod_{i=1}^m (x_i^*)^{\beta_{ji}}$. The parameter κ in the feedback equation (9) can be always (for any controller parameter k_γ) chosen such that the equilibrium point u^* is a given positive value, and satisfies

$$0 = \kappa + \sum_{j=1}^{m} k_j q_j^* + k_\gamma u^* \tag{11}$$

The equilibrium point of u will shift the equilibrium point of the original system to a new one $\overline{x}^*$ that satifies

$$-\lambda_i = \sum_{j=1}^{m} \alpha_{ij} \overline{q}_j^* + \gamma_i u^*, \quad i = 1, \ldots, n \tag{12}$$

but now $\overline{q}_j^* = \prod_{i=1}^m (\overline{x}_i^*)^{\beta_{ji}}$. In order to ensure that the shifted equilibrium point remains in the positive orthant, one should add linear constraints (upper bounds) on the controller parameters $\gamma_i,\ i = 1, ..., n$, when these parameters are determined by solving the stabilizing controller design LMI. These upper bounds will depend on the chosen u^*.

A possible solution for the problem of equilibrium point shifting is to introduce an additional static QP state feedback [16].

B. Decomposition of the closed-loop parameter matrices

The representing closed-loop parameter matrices $\tilde{A}$ and $\tilde{B}$ have now the following form

$$\tilde{B} = \left[\begin{array}{cc} B^* & \underline{0} \\ \underline{0}^T & 1 \\ \hline \overline{B} & \underline{0} \end{array} \right], \tag{13}$$

$$\tilde{A} = \left[\begin{array}{cccc|ccc} & A^* & & \begin{array}{c}\gamma_1 \\ \vdots \\ \gamma_n\end{array} & & \overline{A} & \\ k_1 & \ldots & k_n & k_\gamma & k_{n+1} & \ldots & k_m \end{array} \right] \tag{14}$$

Then we can easily recognize the invertible blocks $\tilde{B}^*$ of $\tilde{B}$ and $\tilde{A}^*$ of $\tilde{A}$, that are

$$\tilde{B}^* = \begin{bmatrix} B^* & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}, \quad \tilde{B}^{*-1} = \begin{bmatrix} B^{*-1} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}, \tag{15}$$

$$\tilde{A}^* = \begin{bmatrix} & A^* & & \underline{\gamma} \\ k_1 & \ldots & k_n & k_\gamma \end{bmatrix}, \quad \underline{\gamma} = \begin{bmatrix} \gamma_1 \\ .. \\ \gamma_n \end{bmatrix}. \tag{16}$$

C. The dynamically similar reduced linear dynamics

In order to stabilize the closed-loop system, its locally dynamically similar linear counterpart is constructed first using the X-factorable transformation described in sub-section II-B.

The parameter matrix $\tilde{M}_{red}^* \in \mathbb{R}^{n+1 \times n+1}$ of the reduced dynamically similar ODE can be obtained by using the closed-loop parameter matrices $\tilde{A}$ and $\tilde{B}$ from (14) and (13) substituted into (7) as

$$\tilde{M}_{red}^* = \begin{bmatrix} & M_{red}^* & & B^* \underline{\gamma} \\ k_1 & \ldots & k_n & k_\gamma \end{bmatrix} \tag{17}$$

Here again we note, that the linear ODE with parameter matrix $\tilde{M}_{red}^*$ is only locally dynamically similar to the original closed-loop system in a neighborhood of the equilibrium point of interest .

The following properties of $\tilde{M}_{red}^*$ are important from the stabilizing controller design purposes

(i) The matrix $\tilde{M}_{red}^*$ depends *linearly* on the controller parameters $k_1, .., k_n, k_\gamma$.
(iii) It does not depend on the rest of the controller parameters $k_{n+1}, .., k_m$ (recall, that $m \geq n$).
(iii) The matrix $\tilde{M}_{red}^*$ depends *linearly* on the input coefficient parameters $\gamma_1, .., \gamma_n$.
(iv) One can easily check the stability of the closed loop system with controller parameters $k_1, .., k_n, k_\gamma$ and input coefficient parameters $\gamma_1, .., \gamma_n$.

IV. STABILIZING DYNAMIC FEEDBACK DESIGN

The constant feedback gains in (9) will be chosen in order to stabilize the closed-loop system. These parameters are collected in the following vectors

$$\begin{gathered} k = \begin{bmatrix} k_1 & \ldots & k_n & k_\gamma \end{bmatrix}^T, \\ k_e = \begin{bmatrix} k_1 & \ldots & k_n & k_\gamma & k_{n+1} & \ldots & k_m \end{bmatrix}^T \\ \gamma = \begin{bmatrix} \gamma_1 & \ldots & \gamma_n \end{bmatrix}^T, \\ . \end{gathered} \tag{18}$$

A. Locally stabilizing feedback

In order to obtain a locally stabilizing dynamic feedback controller (9), we can make the reduced dynamically similar ODE of the closed-loop system stable, because this system and the original one will be both stable in the neighborhood of a chosen equilibrium point. Equation (17) shows that the matrix $\tilde{M}_{red}^*$ depends linearly on k and γ in (18). Therefore we can choose a positive definite symmetric matrix $\tilde{P}$ ($\tilde{P} = \tilde{P}^T$, $\tilde{P} \succ 0$), and solve the following LMI for k and γ

$$\tilde{P}\, \tilde{M}_{red}^* + (\tilde{M}_{red}^*)^T\, \tilde{P} \prec 0. \tag{19}$$

1) Stability region: The domain of attraction, i.e. the stability region of the locally stabilizing controller can in principle be estimated by solving an LMI for its corner points [20], [21] using the fact, that the matrix $\tilde{P}$ gives rise to a quadratic Lyapunov function in the reduced linear state space. However, further studies are needed to take into account the effect of the non-invertible similarity transformation between the original autonomous QP system and its reduced linear counterpart.

B. Globally stabilizing feedback

The above locally stabilizing dynamic feedback design shows, that one has some degrees of freedom to use for making this design globally stabilizing, too. For this purpose we can use the full feedback vector k_e in (18) together with appropriately choosing the parameter matrix $\tilde{P}$. We aim at finding k_e such that

$$C\, \tilde{M}(k_e, \gamma) + \tilde{M}^T(k_e, \gamma)\, C \prec 0 \tag{20}$$

for a diagonal matrix C with positive elements, i.e. $C = \text{diag}\{c_1, ..., c_{m+1}\},\ c_i > 0$.

From the decomposition of the parameter matrices $\tilde{A}$ and $\tilde{B}$ from (14) and (13), and from $\tilde{M} = \tilde{B}\,\tilde{A}$ we obtain

$$\tilde{M} = \left[\begin{array}{ccccccc} & B^*A^* & & B^*\underline{\gamma} & & B^*\overline{A} & \\ k_1 & \dots & k_n & k_\gamma & k_{n+1} & \dots & k_m \\ & \overline{B}A^* & & \overline{B}\underline{\gamma} & & \overline{BA} & \end{array}\right]. \tag{21}$$

This shows that $\tilde{M}$ *depends linearly on the controller parameters* k_e *and* γ. Therefore, with a fixed C, (20) is a linear matrix inequality with unknowns k_e and γ. We remark that if (20) is fulfilled for a positive diagonal C, then $\tilde{M}$ is called *diagonally stabilizable*. Diagonal stabilizability is a much more severe condition than the solvability of a classical Lyapunov equation like (19) with a non-diagonal P matrix [14], and algebraically it has only been characterized for at most 3×3 matrices. Since k_e and γ contain altogether $m + n + 1$ parameters, it can be expected that the chance for the solvability of (20) is generally decreasing when the problem size (i.e. n) is increasing, but special system structures can often be used for assuring and proving diagonal stabilizability [14]. It is a straightforward idea to search k_e, γ and C parallelly that leads to a BMI problem (known to be NP-hard), but we will not elaborate on this in the current paper.

V. CASE STUDIES

The above feedback design methods are illustrated on nonlinear chemical reaction network examples.

A. Stabilizing dynamic feedback of the Brusselator dynamics

1) Brusselator: The results of section IV are illustrated on a chemical reaction network called Brusselator [22] which is described by the set of ODEs (22)

$$\begin{aligned} \dot{x}_2 &= (k_{12}\,x_1^* + k_{32}\,x_3^*) + (-k_{21} - k_{23} - k_{67}\,x_5^*)x_2 - \\ &\quad -k_{54}\,x_2^3 + k_{45}\,x_2^2 x_4 + k_{76}\,x_6^*\,x_4 + \gamma_1\,x_2\,(u - u^*) \\ \dot{x}_4 &= -k_{45}\,x_2^2\,x_4 + k_{54}\,x_2^3 + k_{67}\,x_5^*\,x_2 - k_{76}\,x_6^*\,x_4 \\ &\quad +\gamma_2\,x_4\,(u - u^*) \end{aligned} \tag{22}$$

The parameter values $k_{ij} = 1,\ \forall i, j,\ u^* = 10,\ x_1^* = x_3^* = 1,\ x_5^* = 16$, and $x_6^* = 0.5$ (with $u = 10$) result in an oscillating behavior of the open loop system (22) around the positive equilibrium $[x_2^*,\ x_4^*]^T = [1,\ 11\frac{1}{3}]^T$. It is apparent, that the system is originally in the form (8).

Let us introduce the dynamic feedback law of the form (9)

$$\begin{aligned} \dot{u} &= u\left(\kappa + k_1\,x_2^2 + k_2\,x_2\,x_4 + k_3\,x_2^{-1} + k_4 x_2^{-1} x_4 + \right. \\ &\quad \left. +k_5 x_2^3\,x_4^{-1} + k_6 x_2\,x_4^{-1} + k_\gamma\,u\right) \end{aligned} \tag{23}$$

with the following decomposed exponent and coefficient matrices

$$\tilde{B} = \left[\begin{array}{ccc} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\tilde{A} = \left[\begin{array}{ccc|cccc} -1 & 1 & \gamma_1 & 2 & \frac{1}{2} & 0 & 0 \\ -1 & 0 & \gamma_2 & 0 & 0 & 1 & 16 \\ k_1 & k_2 & k_\gamma & k_3 & k_4 & k_5 & k_6 \end{array}\right]$$

The corresponding parameter matrix $\tilde{M}_{red}^*$ of the underlying reduced linear dynamics depends linearly on the control parameters k and γ as follows.

$$\tilde{M}_{red}^* = \begin{bmatrix} -5 & 3 & 2\,\gamma_1 \\ 14.5 & -15.5 & \gamma_1 + \gamma_2 \\ k_1 & k_2 & k_\gamma \end{bmatrix}$$

2) Locally stabilizing feedback design: For the locally stabilizing feedback design, it is enough to solve the Lyapunov inequality (19) for symmetric matrices (see Section IV-A). In our case the positive definite symmetric matrix $\tilde{P}$ with randomly chosen integer elements is

$$\tilde{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

for which the LMI (19) can be solved. (Note that the basic principle of generating $\tilde{P}$ was to extend an initial positive definite diagonal matrix with random off-diagonal elements preserving positivity.) For example,

$$\begin{aligned} k &= \begin{bmatrix} 2 & -4.4963 & -3.4963 \end{bmatrix} \\ \gamma &= \begin{bmatrix} -2 & 9.9925 \end{bmatrix} \end{aligned} \tag{24}$$

is a feasible solution, which means, that (24) locally stabilizes the system. Figure 1 shows a trajectory of the closed loop system with $\kappa = 0$ evolving to the strictly positive, locally stable steady state $x^* = [1.529,\ 0.295,\ 0.758]^T$. The dynamics of the underlying reduced linear system centered to x^* is also presented in Figure 1.

After setting $u(0) = 0.758$ (i.e. to its steady state value), a simulation-based analysis was performed to explore the domain of attraction (DOA) of the closed loop system. The controlled system was simulated from different initial values obtained by gridding a part of the nonnegative orthant of the (x_1, x_2) space. All the solutions from the box $([0, 1000], [0, 1000])$ were found to converge to x^*, and no such point in the nonnegative part of the state space was found (even outside of this box) that did not belong to the DOA corresponding to x^*. This result suggests that the restrictive condition of diagonal stabilizability is often not necessary for the (global) stability of the closed loop QP system.

3) Globally stabilizing feedback design: The LMI (20) has been also formulated for the Brusselator closed loop dynamics using

$$\tilde{M} = \tilde{B}\,\tilde{A}$$

however, the LMI could not be solved for any diagonal Lyapunov function parameter matrix C. This means, that the global stability of the system (although it was expected from the DOA analysis) could not be proven using the selected Lyapunov function family.

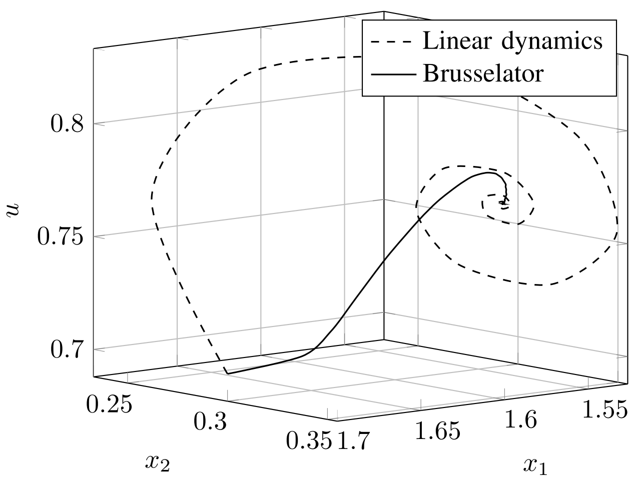

Fig. 1. Some trajectories of the underlying linear dynamics (dashed) and the Brusselator dynamics.

B. Stabilizing dynamic feedback design for a Lotka-Volterra model

The presented stabilizing feedback design methods is represented using a Lotka-Volterra model, which is widely used in population dynamics [19]. It is important to note, that the dynamics of the quasi-monomials q_i of a QP system form a Lotka-Volterra system. Moreover, it is easy to see, that Lotka-Volterra systems are special QP systems with $B = I$.

1) Lotka-Volterra model: Consider the Lotka-Volterra population dynamics defined by the following equation

$$\begin{aligned} \dot{x}_1 &= x_1\,(8 - 2x_1 - 3x_2 - 2x_3 - 0.1x_4 + \gamma_1 u) \\ \dot{x}_2 &= x_2\,(7 - x_1 - 2x_2 - x_3 + \gamma_2 u) \\ \dot{x}_3 &= x_3\,(8 - x_1 - 2x_2 - 2x_3 + \gamma_3 u) \\ \dot{x}_4 &= x_4\,(1.1 - 0.1x_3 - x_4) \end{aligned} \tag{25}$$

The above system has no equilibrium in the strictly positive orthant, i.e. the Lyapunov function in (5) cannot be used for stability analysis in the open loop case.

2) Locally stabilizing dynamical feedback design: The control aim is to design a dynamical controller that locally stabilizes the system around a positive equilibrium point. The feedback law is given in the form (9), i.e.

$$\dot{u} = u\,(\kappa + k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 + k_\gamma u) \tag{26}$$

The closed loop QP (actually, Lotka-Volterra) system can be described by the following parameter matrices:

$$B = B^* = \mathrm{diag}\{1,\ 1,\ 1,\ 1,\ 1\}$$

$$\tilde{M}^*_{red} = \tilde{M} = A^* = A = \begin{bmatrix} -2 & -3 & -2 & -0.1 & \gamma_1 \\ -1 & -2 & -1 & 0 & \gamma_2 \\ -1 & -2 & -2 & 0 & \gamma_3 \\ 0 & 0 & -0.1 & -1 & 0 \\ k_1 & k_2 & k_3 & k_4 & k_\gamma \end{bmatrix}$$

The selection of the positive definite symmetric matrix P has been made randomly (see Section V-A.2) using some heuristics from the feedback structure. The choice

$$P = \begin{bmatrix} 54 & -76 & 3 & -24 & 0 \\ -76 & 178 & 1 & 32 & 0 \\ 3 & 1 & 18 & 11 & -8 \\ -24 & 32 & 11 & 338 & -18 \\ 0 & 0 & -8 & -18 & 18 \end{bmatrix}$$

resulted in a feasible solution for the locally stabilizing feedback design LMI:

$$\begin{aligned} k &= \begin{bmatrix} 0 & 0 & 0 & -11.0120 & -13.6700 \end{bmatrix} \\ \gamma &= \begin{bmatrix} 0 & 0 & -9.9878 \end{bmatrix}. \end{aligned} \tag{27}$$

Using the control parameter $\kappa = -2.34$, the system could be shifted to the following stable positive equilibrium point

$$x^* = [6.2,\ 6.1,\ 1,\ 1,\ 0.171]^T.$$

3) Globally stabilizing dynamical feedback: The control aim is to design a dynamical controller that globally stabilizes the system in the positive orthant. The feedback law is the same as in the previous case (26). The parameter matrix of the corresponding LMI (20) is:

$$\tilde{M} = A^* = A = \begin{bmatrix} -2 & -3 & -2 & -0.1 & \gamma_1 \\ -1 & -2 & -1 & 0 & \gamma_2 \\ -1 & -2 & -2 & 0 & \gamma_3 \\ 0 & 0 & -0.1 & -1 & 0 \\ k_1 & k_2 & k_3 & k_4 & k_\gamma \end{bmatrix}$$

The globally stabilizing feedback design LMI (20) could not be solved for $C = I$. So in this case the parameter matrix C of Lyapunov function (5) has been introduced as extra variables in (20). Of course, the problem to be solved became a bilinear matrix inequality with two sets of variables one set containing the controller parameters and another constructed from the diagonal elements of C. A possible feasible solution for the BMI (28) is the following one

$$\begin{aligned} k &= \begin{bmatrix} 1 & -1 & -1 & 0 & -12.6 \end{bmatrix} \\ \gamma &= \begin{bmatrix} -1 & -2 & -3 \end{bmatrix} \\ C &= \mathrm{diag}\{1,\ 1.97,\ 4.6,\ 1,\ 2.3\}, \end{aligned} \tag{28}$$

which means, that the global stability of the closed loop system using control parameters k and γ can be proven using the Lyapunov function (5) with parameter matrix C.

It is important to note, that the feedback with parameters (28) globally stabilizes the system, only if there exists an equilibrium point of the closed loop system in the positive orthant.

Using (11) the choice $\kappa = 13.6$ ensures, that the closed loop system has a globally asymptotically stable equilibrium point in

$$x^* = [1,\ 1,\ 1,\ 1,\ 1]^T. \tag{29}$$

VI. CONCLUSION AND FUTURE WORK

In this paper a simple specially parametrized dynamic QP feedback controller is proposed for locally stabilizing a QP system. Based on the underlying dynamically similar linear system of a QP system [7], the dynamic QP feedback controller is designed to stabilize the closed-loop system using a pre-defined quadratic control Lyapunov function.

It has been shown, that the parameter matrix of the dynamically similar reduced linear dynamics depends linearly on the feedback gain parameters, this way the stabilizing feedback controller design problem is equivalent to a linear matrix inequality. With the positive definite symmetric parameter matrix of the quadratic Lyapunov function fixed, the controller parameters can be computed by solving an LMI. The domain of attraction of the closed loop system is also investigated. We found that the usual diagonal stability condition for the global stability of the closed-loop system may be too restrictive.

Further work includes to develop a domain of attraction analysis method for the locally stabilized closed-loop system. The relaxation of the diagonal stability condition for global stability will also be investigated. The selection of the feedback structure will also be improved based on the structural properties of the underlying linear dynamics and the dynamically similar chemical reaction network realization [7] of the system.

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