LQ Control of Lotka-Volterra Systems Based on their Locally Linearized Dynamics ⋆,⋆⋆

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Abstract: This work applies the LQ control framework to the class of quasi-polynomial and Lotka-Volterra systems through the linearized version of their nonlinear system model. The primary aim is to globally stabilize the original system with a suboptimal LQ state feedback by means of a well-known entropy-like Lyapunov function that is related to the diagonal stability of linear systems. This aim can only be reached in the case when the quasi-monomial composition matrix is invertible. In the rank-deficient case only the local stabilization of the system is possible with an LQ controller that is designed using the locally linearized model of the closed-loop system model.

Keywords: Lyapunov stability, nonlinear systems, LQ control, linear matrix inequalities

1. INTRODUCTION

A wide range of nonlinear systems can only be tackled using nonlinear techniques (Isidori (1995)). The majority of such techniques are applicable only for a narrow class of nonlinear systems, while the more generally applicable methods suffer from computational complexity problems. One possible way of balancing between general applicability and computational feasibility is to find nonlinear system classes with good descriptive power but well characterized structure, and utilize this structure when developing control design methods. This is possible, for example, in the case of quasi-polynomial systems, that is the subject of this paper.

Previous work in the field of quasi-polynomial systems include the paper of Figueiredo et al. (2000), which gives a sufficient condition for the global stability of quasi-polynomial systems in terms of the feasibility of a linear matrix inequality (LMI). Based on this result, it has been shown in Magyar et al. (2008), that the globally stabilizing state feedback design for quasi-polynomial systems is equivalent to a bilinear matrix inequality. It is also shown there, that although the solution of a bilinear matrix inequality is an NP hard problem, an iterative LMI algorithm could be used. A summary of linear and bilinear matrix inequalities and the available software tools for solving them can be found in VanAntwerp and Braatz (2000).

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Another control synthesis algorithm for polynomial systems is presented in Tong et al. (2007). A different approach has been presented in Magyar and Hangos (2015) where Lotka-Volterra models has been globally stabilized based on their underlying linear model.

The aim of this paper is to apply a *LQ based state feedback controller* for quasi-polynomial and Lotka-Volterra systems through a locally linearized model corresponding to a (unique) positive equilibrium point of the closed-loop system. The primary aim is the formulation of a LQ problem that yields a diagonally stable LTI system and the corresponding globally asymptotically stable Lotka-Volterra or quasi-polynomial system. Of course, the case when the quasi-monomial composition matrix is rank-deficient, it is far from being trivial and one can expect only local asymptotic stability in this case.

2. BASIC NOTIONS

The most important results on quasi-polynomial (QP) and Lotka-Volterra (LV) systems and on their stability analysis are briefly presented here.

2.1 Quasi-Polynomial and Lotka-Volterra Systems

The system dynamics of an *autonomous quasi-polynomial* (QP) system can be described by a set of differential-algebraic equations (DAEs), where the ordinary differential equations

$$\frac{dz_i}{dt} = z_i \left(\lambda_i + \sum_{j=1}^{m} \alpha_{ij} q_j \right) \ , \ i = 1, ..., n, \qquad (1)$$

are equipped by the so called quasi-monomial relationships

$$q_j = \prod_{i=1}^{n} z_i^{\beta_{ji}}, \qquad (2)$$

that are apparently nonlinear (monomial-type) algebraic equations. Two sets of variables are defined, that are (i) the differential variables z_i, $i = 1, ..., n$, and (ii) the quasi-monomials q_j, $j = 1, ..., m$. The parameters of the above model are collected in the coefficient matrix $[A]_{ij} = \alpha_{ij}$, *quasi-monomial composition matrix* $[B]_{ji} = \beta_{ji}$ and a vector $[\lambda]_i = \lambda_i$. Then equation (1) can be written in the compact form

$$\dot{z} = D(z)\,(\lambda + A\,q)\,, \qquad (3)$$

where $D(\cdot)$ stands for $diag(\cdot)$.

It is easy to see that Lotka-Volterra systems form a special subset of the quasi-polynomial systems with the choice $B = I$, and thus $q = z$ with $n = m$

$$\dot{z} = D(z)\,(\lambda + A\,z)\,. \qquad (4)$$

This constitutes a special square invertible case for the quasi-monomial composition matrix B.

Lotka-Volterra form It can be shown (see Hernández-Bermejo and Fairén (1995)) that the class of QP systems is closed under the so called quasi-monomial transformation (QM transformation), where the product $M = BA$ remains constant when transforming a QP model. This way the QM transformation splits the set of QM models into equivalence classes that are represented by a Lotka-Volterra model where the differential variables are the quasi-monomials

$$\dot{q} = D(q)(B\,\lambda + BAq) = D(q)(B\,\lambda + Mq), \qquad (5)$$

where q satisfy the algebraic equations (2).

We can consider the logarithm of these algebraic equations because of the positivity of the two sides

$$\underline{\ln}\ q = B \cdot \underline{\ln}\ z, \qquad (6)$$

where $[\underline{\ln}\ x]_i = \ln\ x_i$. Then (6) is equivalent to

$$\underline{\ln}\ q \in \mathrm{range}(B). \qquad (7)$$

This manifold (7) is an invariant subspace of the dynamics (5) because

$$\frac{d\underline{\ln}\ q}{dt} = B\,(\lambda + Aq) \in \mathrm{range}(B). \qquad (8)$$

It is easy to see that when the matrix B is invertible then

$$z = \exp(B^{-1}\,\underline{\ln}\ q) \qquad (9)$$

for all $q \in \mathbb{R}^n_{>0}$. It means that the algebraic equation (2) has a positive solution for all $q \in \mathbb{R}^n_{>0}$.

In the usual case of $m > n$, the right side of the transformed ODE (5) would be simpler, but we have to consider the algebraic conditions (2).

Steady-state points The non-zero steady-state point(s) of the dynamic equations (1) are obtained by setting the left-hand sides equal to zero, and solve the equations

$$\mathbf{0} = \lambda + A \cdot q^*, \qquad (10)$$

for q^* (the vector q^* has a quasi-monomial relationship with the equilibrium point z^*). *Generally, this equation has a unique solution if* A is quadratic and invertible, but the solution is not necessarily positive.

Otherwise, if $m > n$, then the set of equations (10) may have infinitely many solutions. However, the set of algebraic equations (6) puts a set of nonlinear constraints to the elements of the vector q^* (i.e. the vector q should be taken from a lower dimensional manifold of the quasi-monomial space) that may result in a unique equilibrium point even in this case. The existence of strictly positive solutions without algebraic constraints can be tested by various algorithms or simple linear programming.

Quasi-Polynomial models with input Let us consider a linear input structure for the original QP model (1), that can be formally derived by regarding λ as a function of the input vector u

$$\lambda = \phi\,u, \quad u \in \mathbb{R}^p, \quad \phi \in \mathbb{R}^{n\times p}, \quad p \le n \qquad (11)$$

such that the state equation is in the form

$$\dot{z} = D(z)\,(\phi\,u + A\,q) \qquad (12)$$

with the algebraic equations (6).

2.2 Stability Condition of QP systems

Assume that there exists a positive steady-state point z^* of the QP system (1). Then this steady state point is globally asymptotically stable if there exists a positive diagonal matrix P for the product matrix $M = BA$ such that

$$MP + PM^T < 0, \qquad (13)$$

or

$$QM + M^TQ < 0, \qquad (14)$$

where $Q = P^{-1}$ is positive definite diagonal matrix (Gléria et al. (2001); Figueiredo et al. (2000)). In this case, the matrix M is called diagonally stable (Kaszkurewicz and Bhaya (2012)).

It is important to note that the feasibility of the above LMI is a sufficient (but not always necessary) condition for the stability, as it is derived from the dissipativity property of the entropy-like Lyapunov function

$$V(q) = \sum_{i=1}^{m} \gamma_i \left(q_i - q_i^* - q_i^* \ln \frac{q_i}{q_i^*} \right)$$

where $\gamma_i > 0$ and $Q = D(\gamma)$.

2.3 Locally Linearized QP Model

The linearized version of the QP model (1) around its positive equilibrium point z^* is in the form

$$\frac{\Delta z}{dt} = \left[D(z^*)\,A\,D(q^*)\,B\,D(z^*)^{-1}\right]\Delta z \qquad (15)$$

where $\Delta z = z - z^*$. When the matrix B is invertible, then we can transform (15) with the linear transformation

$$x = \left[D(q^*)\,B\,D(z^*)^{-1}\right]\Delta z = T\Delta z \qquad (16)$$

with the transformation matrix

$$T = D(q^*)\,B\,D(z^*)^{-1}\,, \qquad (17)$$

and the transformed system is in the form

$$\dot{x} = D(q^*)\,B\,A\,x = M^*x. \qquad (18)$$

2.4 Diagonal Stability of the QP and Linearized-QP Models

The following lemma states the equivalence of the diagonal stability of the original QP and the transformed linearized system (18).

Lemma 1 The matrix M^* is diagonally stable if and only if the matrix M is diagonally stable.
Proof: =>: If the matrix M^* is diagonally stable then there exists a positive diagonal Q such that $Q\,D(q^*)\,M + M^T\,D(q^*)\,Q < 0$. Then, the matrix M is diagonally stable with the matrix $D(q^*)\,Q$.
<=: If the matrix M is diagonally stable then there exists a positive diagonal Q such that $Q\,M + M^T\,Q < 0$. Then, the matrix M^* is diagonally stable with the diagonal matrix $Q\,D(q^*)^{-1}$.

3. LQ DESIGN PROBLEM FOR QP-SYSTEMS

In this section, two feedback design methods for quasi-polynomial and Lotka-Volterra systems are presented. The first method achieves diagonal stability and improves the local LQ performance. This method assumes an invertible quasi-monomial composition matrix B. The second method will only locally stabilize the system with local LQ performance, but it assumes only a full rank quasi-monomial composition matrix B with linearly independent column vectors.

3.1 Feedback Structure

Let us consider the nonlinear state feedback in the form

$$u = K\,q + u^* \tag{19}$$

where u^* is a constant that moves the closed loop equilibrium to the desired steady-state point z^*.

3.2 The Steady-state Point of the Closed Loop System

Let us consider the steady state equations (10) in the closed loop case

$$\phi\,(K\,q^* + u^*) + A\,q^* = \mathbf{0}. \tag{20}$$

Let

$$u^* = -K\,q^* + u^{**} \tag{21}$$

then we can set the positive vector q^* to a steady state of the closed loop system if there exists u^{**} such that

$$\phi\,u^{**} + A\,q^* = \mathbf{0}. \tag{22}$$

3.3 Linearized Closed Loop System and Input

When the positive vector z^* is a suitable steady state of the closed loop system, then the linearization of the closed loop QP system is

$$\frac{d\Delta z}{dt} = \left(D(z^*)\,[A + \phi K]\,D(q^*)BD(z^*)^{-1}\right)\Delta z. \tag{23}$$

In that case the linearized input is

$$u' = KD(q^*)BD(z^*)^{-1}\Delta z. \tag{24}$$

3.4 Suboptimal LQ with Diagonal Stability

In this subsection, LQ control with diagonal stability for QP system is considered. For this, *we assume that B is invertible, i.e. it is a full rank square matrix with $m = n$.*

When the matrix B is invertible, the linearized closed-loop system (23) can be transformed into the following form

$$\dot{x} = (M^* + N^*K)x \tag{25}$$

using the transformation matrix T in equation (17), where $M^* = D(q^*)\,B\,A$ and $N^* = D(q^*)\,B\,\phi$.

As stated in the Lemma 1, when the state matrix of the linearized and transformed closed-loop system

$$M^* + N^*K \tag{26}$$

is diagonally stable, then the corresponding QP system is diagonally stable, too.

If the pair $(M^*,\ N^*)$ is diagonally stabilizable, then a suboptimal LQ of the linearized closed loop system with diagonal stability can be computed by following the method of Haddad et al. (2009).

Let the LQ objective function of the closed-loop linearized QP system be

$$J(K) = \int_0^\infty \|Q^{\frac{1}{2}}\Delta z\|_2^2 + \|R^{\frac{1}{2}}KD(q^*)BD(z^*)^{-1}\Delta z\|_2^2\,dt \tag{27}$$

where Q and R are positive definite matrices with the appropriate dimensions, that can be chosen arbitrarily to satisfy our control performance aims. To guarantee diagonal stability, we have to design the feedback to the transformed and linearized system. Then, the transformed LQ objective is in the form

$$J(K) = \int_0^\infty \|Q^{\frac{1}{2}}T^{-1}x\|_2^2 + \|R^{\frac{1}{2}}Kx\|_2^2\,dt \tag{28}$$

using the transformation matrix T in equation (17). The suboptimal feedback gain K can be computed by solving the following LMI problem

$$\min_{P,X,Y} \operatorname{Tr}(T^{-T}QT^{-1}P) + \operatorname{Tr}(X) \tag{29}$$

subject to

$$M^*\,P + P\,(M^*)^T + N^*\,Y + Y^T\,(N^*)^T + I < 0 \tag{30}$$

$$\begin{bmatrix} X & R^{1/2}Y \\ Y^T R^{1/2} & P \end{bmatrix} > 0 \tag{31}$$

where P is a *positive diagonal matrix*, X is a positive definite matrix and $Y = KP$.

The resulted closed-loop system (25) is diagonally stable and has suboptimal LQ performance such that

$$J(K_{\text{opt}}) \le J(K) \le \operatorname{Tr}(T^{-T}QT^{-1}P) + \operatorname{Tr}(X) \tag{32}$$

where K and K_{opt} are the solution of the problem (29) - (31) with and without the diagonal restriction, respectively.

3.5 Local LQ in the general non-invertible case

When the matrix B is not invertible, then the open-loop QP system can not be diagonally stabilized, because then M^* is also rank deficient. Then we can still *use the nonlinear feedback* (19) *to design an LQ feedback to the linearized system that will only locally stabilize the system.*

Let us consider the linearized closed-loop system (23) with the new notations

$$\frac{\Delta z}{dt} = (A' + B'K')\Delta z, \tag{33}$$

where

$$A' = D(z^*)AD(q^*)BD(z^*)^{-1}, \quad B' = D(z^*)\phi,$$
$$C' = D(q^*)BD(z^*)^{-1}, \quad K' = KC'.$$

Observe, that the matrix C' is the same as the transformation matrix T in equation (17), and can be regarded as a transformed version of the quasi-monomial composition matrix B of the same size, i.e. $C' \in \mathbb{R}^{m\times n}$.

When the rank of matrix C' is n – i.e. the matrix B is of full rank with linearly independent columns – then for all K' there exits a matrix K such that

$$K' = KC'. \tag{34}$$

Therefore, we can solve the LQ problem for the linear system (A', B'), and use the resulted feedback gain K' to compute the real feedback gain in equation (19) from (34).

4. CASE STUDIES

Two simple examples are presented here to demonstrate the LQ controller, one for the square invertible case when globally stabilizing design is possible, and another one for the rank-deficient locally stabilizing design.

4.1 Suboptimal Diagonally Stabilizing LQ Design

Let the open-loop QP system be

$$\begin{aligned} \dot{z}_1 &= z_1(2z_2 + 5z_1^2 + u) \\ \dot{z}_2 &= z_2(-3z_2 - 2z_1^2 - u). \end{aligned} \tag{35}$$

The system matrices are

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \; \phi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

First, we have to choose a desired equilibrium point $z^* = [1\ 3]^T$. It fulfils the condition (22) with $u^{**} = -11$. Then, the linear transformation matrix in equation (17) is

$$T = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix},$$

and the matrices of the transformed system is

$$M^* = \begin{bmatrix} -9 & -6 \\ 4 & 10 \end{bmatrix}, \; N^* = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

This system is diagonally stabilizable. Therefore, we can apply the suboptimal and optimal LQ design, too. Let $Q = I$ and $R = 0.1$. Then, the computed gain matrices are

$$K_{SLQ} = [-2.7427 \;\; -15.6096]$$
$$K_{LQ} = [-3.3351 \;\; -15.6064].$$

The transformed controllers of the QP system are

$$u_{SLQ} = -2.7427z_2 - 15.6096z_1^2 + 12.8376 \tag{36}$$

and

$$u_{LQ} = -3.3351z_2 - 15.6064z_1^2 + 14.6116. \tag{37}$$

The suboptimal control transforms the open-loop system to be diagonally stable, so the QP system will be globally asymptotically stable with the equilibrium point z^*. The optimal control stabilizes the QP system only locally, but the local performance is found to be better than in the suboptimal case. Fig. 1 shows the phase portrait of the two controlled cases with different initial conditions.

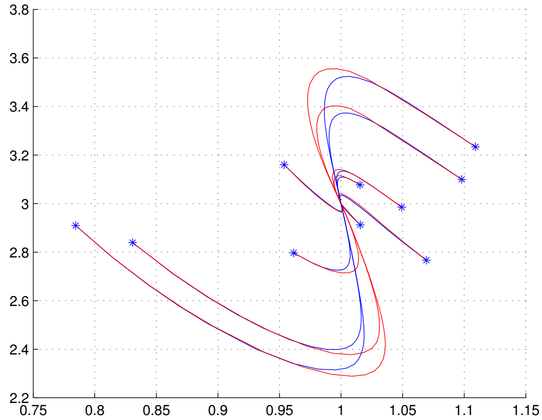

Fig. 1. The phase portrait of the controlled QP system (35) starting from different initial conditions. The blue curves correspond to the diagonally stabilized system, the red ones correspond to the locally stabilized system.

4.2 Local LQ Design

Let us consider the open-loop QP system

$$\begin{aligned} \dot{z}_1 &= z_1(2z_1z_2 + z_1^2 - z_2^2 + 2u) \\ \dot{z}_2 &= z_2(z_1z_2 + z_1^2 - z_2^2 + u). \end{aligned} \tag{38}$$

The system matrices are

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}, \; \phi = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and $q = \begin{bmatrix} z_1z_2 & z_1^2 & z_2^2 \end{bmatrix}^T$.

First, we have to choose a desired equilibrium point $z^* = [1\ 1]^T$. It fulfils the condition (22) with $u^{**} = -1$. In that case, the parameters of the linearized system (33) are

$$A' = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}, \; B' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \; C' = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

In the following, we are going to design a LQ controller for the system (A', B') with the parameters $Q = I$ and $R = 0.1$. The resulted state feedback gain is

$$K' = [-5.4067 \;\; -1.4923]$$

that corresponds to the real control gain

$$K = [0 \;\; -2.7034 \;\; -0.7461].$$

This results in the control input

$$u = -2.7034z_1^2 - 0.7461z_2^2 + 2.4495.$$

Fig. 2 shows the phase portrait of the controlled system with different initial conditions.

5. CONCLUSION

LQ based stabilizing feedback design approaches have been presented in this work for quasi-polynomial and Lotka-Volterra type nonlinear systems. One of the approaches

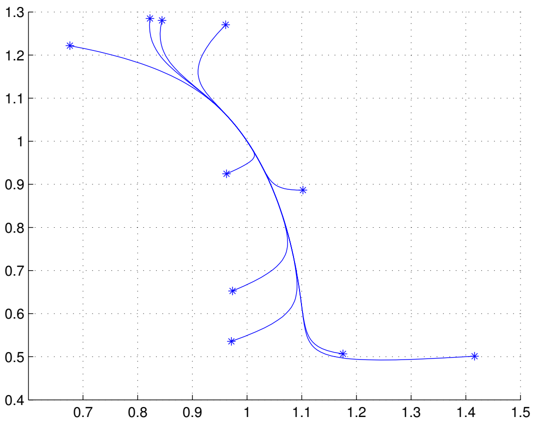

Fig. 2. The phase portrait of the controlled QP system (38) with different initial conditions.

proposes a diagonally stabilizing state feedback design technique based on the linearized version of the quasi-polynomial model, formulated as a set of LMIs. This method yields a suboptimal controller in the LQ sense and can globally stabilize the system in the square invertible $m = n$ case.

In the case of rank deficiency which may be caused by the fact that the Lotka-Volterra model originates from a lower dimensional quasi-polynomial model, only a locally stabilizing LQ controller can be designed using the linearized form of the closed-loop system model.

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