

Chapter 3

Analysis of Qualitative Dynamic Properties of Positive Polynomial Systems using Transformations

Katalin M. Hangos and Gábor Szederkényi

Abstract Two classes of positive polynomial systems, quasi-polynomial (QP) systems and reaction kinetic networks with mass action law (MAL-CRN) are considered. QP-systems are general descriptors of ODEs with smooth right-hand sides, their stability properties can be checked by algebraic methods (linear matrix inequalities). On the other hand, MAL-CRN systems possess a combinatorial characterization of their structural stability properties using their reaction graph.

Dynamic equivalence and similarity transformations applied either to the variables (quasi-monomial and time-reparametrization transformations) or to the phase state space (translated X-factorable transformation) will be applied to construct a dynamically similar linear MAL-CRN model to certain given QP system models. This way one can establish sufficient structural stability conditions based on the underlying reaction graph properties for the subset of QP system models that enable such a construction.

Key words:

3.1 Introduction

The class of positive polynomial ODEs plays an important role in describing the dynamics of physical, chemical and ecological systems, where the positivity of the variables is dictated and ensured by the physical meaning, as

Katalin M. Hangos
University of Pannonia, Veszprém, Hungary; Computer and Automation Research Institute, Budapest, Hungary, e-mail: hangos.katalin@virt.uni-pannon.hu

Gábor Szederkényi
Faculty of Information Technology, Peter Pázmány Catholic University, Budapest, Hungary, e-mail: szederkenyi.gabor@itk.ppke.hu

certain quantities - such as concentrations, pressures, population numbers etc. - cannot take negative values. The dynamic model of such systems may often originate from first physical, chemical or engineering principles - such as conservation - that implies a well defined structure to the right-hand sides of the ODE models.

The notion of positive systems builds upon the *essential nonnegativity* of a function $f = [f_1 \dots f_n]^T : [0, \infty)^n \rightarrow \mathbb{R}^n$, that holds if, for all $i = 1, \dots, n$, $f_i(x) \geq 0$ for all $x \in [0, \infty)^n$, whenever $x_i = 0$ [6]. An autonomous nonlinear system defined on the nonnegative orthant $[0, \infty)^n = \mathbb{R}_+^n \subset \mathcal{X}$

$$\dot{x} = \frac{dx}{dt} = f(x), \quad x(0) = x_0, \quad (3.1)$$

where $f : \mathcal{X} \rightarrow \mathbb{R}^n$ is locally Lipschitz, \mathcal{X} is an open subset of \mathbb{R}^n and $x_0 \in \mathcal{X}$ is nonnegative (or positive) when the nonnegative (or positive) orthant is invariant for the dynamics (3.1). This property holds if and only if f is essentially nonnegative.

The sub-class of quasi-polynomial systems (QP systems in short), to which the well-known Lotka-Volterra equations belong, form a general descriptor class of dynamic systems with smooth nonlinearities in the sense that such systems can be embedded into QP form by adding new auxiliary variables to the system [4]. There exists a parameter-dependent sufficient condition for a given QP system to be globally asymptotically stable [10], that can be checked by solving a linear matrix inequality (LMI).

Deterministic kinetic systems with mass action kinetics or simply chemical reaction networks (CRNs) form a wide class of nonnegative polynomial systems, that are able to produce all the important qualitative phenomena (e.g. stable/unstable equilibria, oscillations, limit cycles, multiplicity of equilibrium points and even chaotic behavior) present in the dynamics of nonlinear processes [2]. The importance of the CRN system class with mass action law (abbreviated as MAL-CRNs) lies in the fact that strong structural (i.e. parameter-independent) stability results exist for the deficiency zero weakly reversible case [7], and recently for the detailed balanced case, when each of the chemical reactions are assumed to be reversible (see [5], [11], [12], [16] and recently [18]).

The aim of this paper is to try to establish a dynamic similarity relationship between the Lotka-Volterra form of QP systems and the linear MAL-CRNs in order to obtain structural stability conditions for the former.

3.2 Quasi-polynomial (QP) systems

The most general class of positive polynomial systems is the class of quasi-polynomial (QP) ones, that are time-dependent autonomous ODEs (3.1)

evolving in the positive orthant \mathbb{R}_+^n , i.e. $x(t) \succ 0$ (element-wise) for $t \geq 0$ and $x_0 \succ 0$.

3.2.1 The ODE form

Two sets of variables are present in the ODE form of a QP system:

- the **state variables** $x_i, i = 1, \dots, n$, and
- the **quasi-monomials (QMs)** $q_j, j = 1, \dots, m$

We assume $m \geq n$.

With these variables the system dynamics is described by an autonomous ODE with quasi-polynomial right-hand sides defined on the positive orthant

$$\frac{dx_i}{dt} = x_i \left(\lambda_i + \sum_{j=1}^m A_{ij} q_j \right), \quad i = 1, \dots, n, \quad (3.2)$$

that is augmented by the following algebraic equations:

$$q_j = \prod_{i=1}^n x_i^{B_{ji}}, \quad j = 1, \dots, m. \quad (3.3)$$

The above equations (3.3) are the so called quasi-monomial (QM) relationships.

The state space $\mathcal{X} \subseteq \mathbb{R}_+^n$ will also be called phase space in the paper.

Algebraic characterization

The real vector $\lambda \in \mathbb{R}^n$ and matrices $B \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{n \times m}$ are the parameters of a QP system model (3.2)-(3.3).

3.2.2 Quasi-monomial transformation and the Lotka-Volterra canonical form

The so called quasi-monomial transformation is an equivalence transformation on the class of QP-systems that allows to form equivalence classes. These classes can be represented by their member in Lotka-Volterra canonical form [3].

3.2.2.1 The quasi-monomial transformation

The so called quasi-monomial transformation or **QM-transformation** in short introduces new state variables

$$x'_j = \prod_{i=1}^n x_i^{\Gamma_{ji}}, \quad j = 1, \dots, n. \quad (3.4)$$

The parameter of the QM-transformation is the real square invertible matrix $\Gamma \in \mathbb{R}^{n \times n}$.

The parameters A and B of a QP system model are transformed to $A' = \Gamma^{-1}A$ and $B' = B\Gamma$, therefore the product $M = BA$ is invariant under the QM-transformation.

3.2.2.2 Lotka-Volterra (LV) canonical form

Being an equivalence transformation, the QM-transformation splits the set of QP models into equivalence classes. **From any QP-model** (3.2)-(3.3) with parameters (A, B, λ) of an equivalence class, the LV model form can be obtained **by QM-transformation and variable extension** such that $B' = I$ with $x' = q$. Then the transformed matrix A' becomes

$$A' = M = B \cdot A. \quad (3.5)$$

The resulting transformed ODE in LV form

$$\frac{dq_l}{dt} = q_l \left(A_l + \sum_{j=1}^m M_{lj} q_j \right), \quad l = 1, \dots, m \quad (3.6)$$

is a homogeneous bi-linear ODE that describes the dynamics in the monomial space $\mathcal{Q} \subseteq \mathbb{R}_+^m$. However, because of the variable extension and the relationship $m \geq n$, the dynamics lives in a lower n -dimensional manifold of the monomial space \mathcal{Q} .

Steady state points

The non-trivial nonnegative steady state points of the original QP equation (3.2) can be obtained (if they exist) by solving the equation

$$0 = \lambda + A \cdot q^* \quad (3.7)$$

for q^* . It is important to note that the equilibrium point is determined in the monomial space \mathcal{Q} , and then it is transformed back to the state space.

Eq. (3.7) is a linear under-determined equation for the vector $q^* \in \mathbb{R}^m$, but Eq. (3.3) gives $m - n$ algebraic relationships between the elements of q^* , therefore one may have a well posed solution (even if it is not unique). As the monomial space is only a subset of \mathbb{R}^m ($\mathcal{Q} \subseteq \mathbb{R}_+^m$), it may occur that no positive equilibrium point exists. Without further investigations, however, we only consider here the case when **a finite number of positive steady-state points exist** in the state space.

The vector-matrix form

In order to develop a compact vector-matrix form, the following notations are introduced

$$\underline{\ln} q^T = [\ln q_1 \ \dots \ \ln q_m]^T, \quad \underline{diag} q = \begin{bmatrix} q_1 & 0 & \dots & 0 & 0 \\ 0 & \cdot & q_i & \cdot & 0 \\ 0 & 0 & \dots & 0 & q_m \end{bmatrix}. \quad (3.8)$$

Then the dynamics (3.6) of a Lotka-Volterra system with a positive steady state point q^* can be written in the following form:

$$\frac{d \underline{\ln} q}{dt} = M \cdot (q - q^*) \quad \text{or} \quad \frac{dq}{dt} = \underline{diag} q \cdot M \cdot (q - q^*). \quad (3.9)$$

3.2.3 The time-rescaling transformation

The so called time-rescaling transformation [9] maps a QP system model to another QP system model in the following way. Let us introduce a transformation vector $\Omega = [\Omega_1 \ \dots \ \Omega_n]^T \in \mathbb{R}^n$, that is used to "rescale" the time in a state-dependent way

$$dt = \prod_{k=1}^n x_k^{\Omega_k} dt'.$$

Then the original QP model (3.2) with parameters (A, B, λ) is transformed to the model that is also in QP-form

$$\frac{dx_i}{dt'} = x_i \sum_{j=1}^{m+1} [\tilde{A}]_{i,j} \prod_{k=1}^n x_k^{[\tilde{B}]_{j,k}}, \quad i = 1, \dots, n, \quad (3.10)$$

where the new parameters $\tilde{A} \in \mathbb{R}^{n \times (m+1)}$ and $\tilde{B} \in \mathbb{R}^{(m+1) \times n}$ are

$$\begin{aligned}
\tilde{A}_{i,j} &= A_{i,j}, \quad i = 1, \dots, n; \quad j = 1, \dots, m \\
\tilde{A}_{i,m+1} &= \lambda_i, \quad i = 1, \dots, n \\
\tilde{B}_{i,j} &= B_{i,j} + \Omega_j, \quad i = 1, \dots, m; \quad j = 1, \dots, n \\
\tilde{B}_{m+1,j} &= \Omega_j, \quad j = 1, \dots, n.
\end{aligned}$$

Note that the **number of monomials is increased by one**, and the new parameter vector $\tilde{\lambda}$ is zero in the transformed system.

It is important to note that by assuming strictly positive state variables, the time-rescaling transformation is a **similarity transformation**, that leaves the equilibrium points and the stability properties unchanged [9].

3.2.4 Stability condition for QP systems

Thanks to the well characterized structure of QP systems with a positive equilibrium point q^* , an easy-to-check sufficient condition for their global (asymptotic) stability exists [10].

A QP system (3.2)-(3.3) with a positive equilibrium point q^* is globally stable if the the linear matrix inequality

$$M^T C + C M \leq 0 \quad (3.11)$$

is solvable for a positive diagonal matrix C , with $M = BA$. In this case, the matrix M is called **diagonally stable**. (The stability is asymptotic, if the inequality (3.11) is strict.) Given the parameter matrix M of the system, the condition (3.11) can be checked effectively by solving a linear matrix inequality (LMI) [19].

It is important to note that the above condition is derived using the following Lyapunov function candidate:

$$V(q) = \sum_{i=1}^m c_i \left(q_i - q_i^* - q_i^* \ln \frac{q_i}{q_i^*} \right). \quad (3.12)$$

Unfortunately, however, the condition (3.11) is rather conservative. Therefore, one may use time-rescaling of the original QP system model to find a dynamically similar QP system model such that it fulfills (3.11). This, however, requires to solve a bilinear matrix inequality (BMI) [14].

3.3 Chemical Reaction Networks with mass action law

Chemical reaction networks with mass action law (MAL-CRNs in short) form an important special sub-class of positive polynomial systems. Their special structure, that will be described briefly in this section, enables to apply parameter-independent robust conditions for their asymptotic stability.

3.3.1 Formal description

Chemical reaction networks [7] are abstract versions of reaction kinetic models in chemistry and bio-chemistry. They are composed of **irreversible elementary reaction steps** in the form



where A_s , $s = 1, \dots, n$ are the chemical components, while α_{sj} and β_{sl} are the **stoichiometric coefficients** that are always non-negative integers.

The linear combinations of components present on each side of a reaction step are called **complexes**, i.e. $C_j = \sum_{s=1}^n \alpha_{sj} A_s$ ($j = 1, \dots, m$) form the set of complexes.

The dynamics of a MAL-CRN is described by an autonomous ODE with polynomial right-hand side on the positive orthant in the following form

$$\dot{x} = \frac{dx}{dt} = Y \cdot A_k \cdot \varphi(x) \quad (3.14)$$

$$\varphi_j(x) = \prod_{s=1}^n x_s^{\alpha_{sj}}, \quad j = 1, \dots, m, \quad (3.15)$$

where the state vector is composed of the concentrations (these are non-negative quantities) of the components (x_s is the concentration of A_s). The non-negative variables in the vector φ are called **(reaction-)monomials**, they span the monomial space.

It should be emphasized, that - in contrast to the reversible reaction steps traditionally considered in the applied mathematical literature (see [5], [11], [12], [16] and recently [18]) we assume **irreversible reactions** in (3.13). This implies that the reaction rate $r_j = k_{j,l} \varphi_j(x)$ of the reaction (3.13) depends only on the composition of the reactant complex, i.e. on the stoichiometric coefficients α_{sj} , $s = 1, \dots, n$ but not on the coefficients β_{sl} , $s = 1, \dots, n$.

The parameters of the model are the complex composition matrix $Y_{sj} = \alpha_{sj}$ and the reaction matrix A_k :

$$[A_k]_{lj} = \begin{cases} -\sum_{\ell=1, \ell \neq j}^m k_{j,\ell}, & \text{if } l = j \\ k_{j,l}, & \text{if } l \neq j, \end{cases} \quad (3.16)$$

where $k_{l,j} > 0$ is the **reaction rate constant** (a positive number) of the reaction $C_l \rightarrow C_j$.

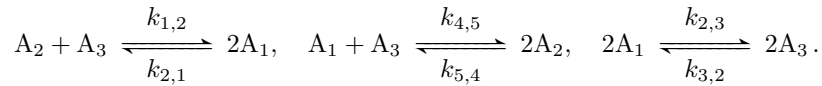
It is important to note that $A_k \in \mathbb{R}^{m \times m}$ is a **Kirchhoff matrix** with zero column sum. Therefore, A_k is rank-deficient.

The reaction structure of a MAL-CRN is described by the so-called **reaction graph**, that is a weighted directed graph. The vertices of the reaction graph correspond to the complexes, and the edges describe reactions that connect the complexes. This means, that a directed edge from the vertex C_j to C_l exists, if there is a reaction $C_j \rightarrow C_l$ in the CRN. The edge weight is the corresponding reaction rate coefficient $k_{j,l}$. Therefore, the Kirchhoff-matrix A_k determines the reaction graph.

It is important to note that the dynamics of CRNs with (generalized) mass conservation evolve in a lower dimensional sub-space of the state space $\mathcal{X} \subseteq \mathbb{R}_+^n$, that is determined by the initial conditions and is called the **stoichiometric compatibility class**.

3.3.1.1 Example: a simple nonlinear MAL-CRN

Let us consider a simple MAL-CRN with the following three reversible reactions



Because the elementary reaction steps are considered irreversible, we break down these reactions into six irreversible steps connecting five complexes (i.e. $m = 5$ with $C = \{A_2 + A_3, 2A_1, 2A_3, A_1 + A_3, 2A_2\}$) with the following reaction-monomials:

$$\varphi(x) = [x_2 x_3 \quad x_1^2 \quad x_3^2 \quad x_1 x_3 \quad x_2^2]^T.$$

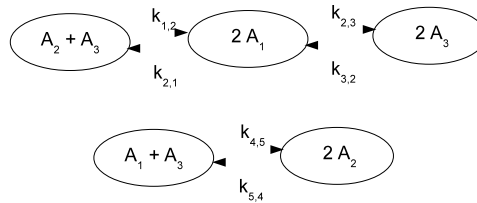


Fig. 3.1 The reaction graph of the example

The dynamic model equations are as follows:

$$\begin{aligned}\dot{x}_1 &= 2k_{1,2}x_2x_3 - (2k_{2,1} + 2k_{2,3})x_1^2 + 2k_{3,2}x_3^2 - k_{4,5}x_1x_3 + k_{5,4}x_2^2 \\ \dot{x}_2 &= -k_{1,2}x_2x_3 + k_{2,1}x_1^2 + 2k_{4,5}x_1x_3 - 2k_{5,4}x_2^2 \\ \dot{x}_3 &= -k_{1,2}x_2x_3 + (k_{2,1} + 2k_{2,3})x_1^2 - 2k_{3,2}x_3^2 - k_{4,5}x_1x_3 + k_{5,4}x_2^2.\end{aligned}$$

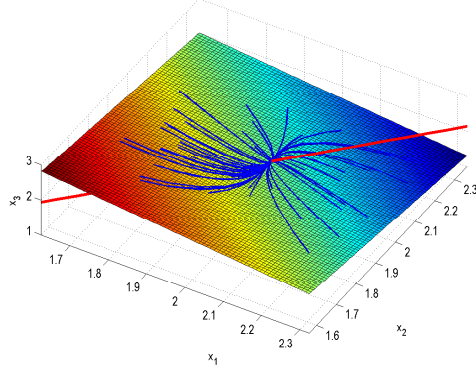


Fig. 3.2 The dynamics of the example evolving in a stoichiometric compatibility class

3.3.2 MAL-CRN structural stability

The **structure** of a MAL-CRN system is determined by the complex composition matrix Y and by its reaction graph (or equivalently its reaction matrix A_k) without its weights, i.e. irrespectively of the actual values of the reaction rate constants.

The structural stability of an ODE can also be defined following this idea.

Definition 3.1. An ODE $\frac{dz}{dt} = F(z, P)$ with parameters P will be called structurally stable with respect to a parameter set \mathcal{P} , if it is stable for every $P \in \mathcal{P}$.

3.3.2.1 MAL-CRN structural properties

The structural properties of a MAL-CRN model are defined based on the graph structure of the reaction graph without its edge weights and on the complex compositions.

A CRN is called **weakly reversible** if whenever there exists a directed path from C_i to C_j in its reaction graph, then there exists a directed path from C_j to C_i . In graph theoretic terms, this means that all components of

the reaction graph are strongly connected components. We shall use the fact known from the literature that a CRN is weakly reversible if and only if there exists a vector b with strictly positive elements in the kernel of A_k , i.e. there exists $b \in \mathbb{R}_+^n$ such that $A_k \cdot b = 0$ [8]. This also implies that the CRN has a positive equilibrium point in the monomial space.

The notion of the **deficiency** of a CRN is built on the set of **reaction vectors** that are defined as: $\mathcal{R} = \{\rho^{(l,k)} = \eta^{(l)} - \eta^{(k)} \mid C_k C_l \in E \text{ in } G\}$, where $\eta^{(i)}$ denotes the i th column of Y . Then the deficiency δ is an integer number that is defined as:

$$\delta = m - \ell - s \quad (3.17)$$

where m is the number of complexes and ℓ is the number of connected components in the reaction graph, while s is the dimension of the stoichiometric sub-space, i.e. $s = \text{rank}(\mathcal{R})$. The zero deficiency property implies the stability of equilibria in a weakly reversible MAL-CRN system.

Deficiency Zero theorem

The **Deficiency Zero Theorem** [7] shows a very robust stability property of a certain class of kinetic systems. It says that deficiency zero weakly reversible networks possess well-characterizable equilibrium points, and independently of the weights of the reaction graph (i.e. that of the system parameters) they are at least locally stable with a known logarithmic Lyapunov function that is also independent of the system parameters. (According to the so-called **Global Attractor Conjecture** that was proved for the single linkage class case in [1], this stability is actually global.)

3.3.3 Linear CRN systems

A linear MAL-CRN is characterized by the equation $Y = I$, that is, $m = n$ and the components form the complexes ($C_i = A_i$, $i = 1, \dots, m$). Then the dynamics is described by the following ODE

$$\frac{dx}{dt} = A_k x \quad (3.18)$$

where A_k is the reaction matrix, that is a Kirchhoff matrix (see Eq. (3.16)). This implies

$$[A_k]_{ii} < 0; \quad [A_k]_{ij} \geq 0, \quad i \neq j; \quad \underline{1} \cdot A_k = \underline{0}, \quad (3.19)$$

where $\underline{1} = [1, \dots, 1]$ is a row vector.

Note that the state and monomial spaces of a linear CRN coincide, and the dynamics is linear in this space.

Because of the $Y = I$ equality, a **linear CRN has always zero deficiency**.

3.4 Transforming LV models to a linear MAL-CRN form

Based on the notion of dynamic similarity and on model transformations we aim at constructing a dynamically similar linear MAL-CRN model to a given Lotka-Volterra model. If this is possible then we can use the structural stability conditions of the linear MAL-CRN model to infer the structural stability of the LV model.

3.4.1 The translated X-factorable transformation

Given an ODE

$$\frac{dz}{dt} = F(z) \quad (3.20)$$

on the positive orthant $z \in \mathbb{R}_+^n$ with $F(z) = 0$.

The **nonlinear translated X-factorable transformation** maps the above ODE into

$$\frac{dz'}{dt} = \underline{diag} z \cdot F(z - z^*), \quad (3.21)$$

where the elements of $z^* = [z_1^*, \dots, z_n^*]^T$ are positive real numbers, and $z = [z_1, \dots, z_n]^T$.

If $F(z)$ is composed of polynomial-type functions with a finite number of singular solutions, then the above transformation can move the singular solutions into the positive orthant, and leaves the geometry of the state (or phase) space unchanged within it (but not at or near the boundary) [17].

The dynamics of the solutions of Eqs. (3.20) and (3.21) are called **structurally similar**.

3.4.2 Constructing a dynamically similar linear CRN form

Let us have a QP system model in its LV form defined on the positive orthant

$$\frac{d\bar{x}}{dt} = \underline{diag} \bar{x} (\Lambda + M\bar{x}) \quad (3.22)$$

with a positive steady state point \bar{x}^* . We want to construct a linear CRN model of the form

$$\frac{d\chi}{dt} = \tilde{A}_k \chi \quad (3.23)$$

such that the two systems are dynamically similar and \tilde{A}_k is a Kirchhoff matrix.

Definition 3.2. Two ODEs are called dynamically similar if they have topologically equivalent state spaces (topologically equivalent phase spaces) [15], and the stability properties of all of their steady state manifolds are the same.

The requirement of dynamic similarity implies that the linear CRN model will also have a positive steady state point χ^* with the same stability property (e.g. globally asymptotically stable).

The construction will be done in two steps. First a dynamically similar linear homogeneous model will be constructed that will be transformed to a CRN model in the second step - if possible.

3.4.2.1 Dynamically similar homogeneous linear system

We observe that MAL-CRN models in Eq. (3.14)-(3.15) form a homogeneous set of equations where the equilibrium point does not appear in the equations. Therefore, we augment the state vector \bar{x} of the model (3.22) by a constant element, then the **homogeneous form** is

$$\frac{dx}{dt} = \underline{diag} x \left(\left[\begin{array}{c|c} M & A \\ \hline 0 & 0 \end{array} \right] x \right), \quad (3.24)$$

where $n = \bar{n} + 1$ and $n \in \mathbb{R}^n$ is the new state vector.

Next we follow the procedures described in [13] by using X-factorable transformation to associate a dynamically similar **linear ODE**

$$\frac{dx}{dt} = \left[\begin{array}{c|c} M & A \\ \hline 0 & 0 \end{array} \right] x = \check{M}x. \quad (3.25)$$

3.4.2.2 Transforming the linear ODE to a potential CRN

In the second step we ensure the zero column sum property of the model by applying a linear state transformation (that is an equivalence transformation of the state spaces) using the invertible transformation matrix

$$T = \left[\begin{array}{c|c} I & 0 \\ \hline -1 \dots -1 & 1 \end{array} \right], \quad T^{-1} = \left[\begin{array}{c|c} I & 0 \\ \hline 1 \dots 1 & 1 \end{array} \right]. \quad (3.26)$$

We apply T to Eq. (3.25) to have $\frac{d\chi}{dt} = \tilde{A}\chi$ with $\chi = Tx$ and

$$\hat{M} = T\check{M} = \left[\begin{array}{c|c} M & \Lambda \\ \hline \underline{m} & \Lambda_n \end{array} \right], \quad \tilde{M} = \hat{M}T^{-1} = \left[\begin{array}{c|c} M + \Lambda \underline{1} & \Lambda \\ \hline \underline{m} + \Lambda_n \underline{1} & \Lambda_n \end{array} \right], \quad (3.27)$$

where $m_i = -\sum_{l=1}^{\bar{n}} [M]_{li}$ and $\Lambda_n = -\sum_{l=1}^{\bar{n}} \Lambda_l$.

The column conservation property holds for both \hat{M} and \tilde{M} .

Finally we obtain that \tilde{M} corresponds to the coefficient matrix \tilde{A}_k of the dynamically similar linear ODE (3.23) that can only be a CRN if \tilde{M} has the required sign patterns in Eq. (3.19) besides of the column conservation property.

3.4.3 Structural stability analysis

The sufficient conditions in the deficiency zero theorem will be used to establish conditions for robust structural stability using the transformed coefficient matrix \tilde{M} in (3.27).

The following properties of the original LV parameter matrices (M, Λ) are needed as sufficient conditions for the structural stability that originate from the required sign pattern property of the CRN coefficient matrix \tilde{A}_k in Eq. (3.19).

1. non-negativity of the parameter vector Λ , i.e.

$$\Lambda_i > 0, \quad i = 1, \dots, \bar{n} \quad (3.28)$$

2. the sign pattern and the strict dominant main diagonal property of M , i.e.

$$M_{ii} < 0, \quad M_{ij} \geq 0, \quad |M_{ii}| \geq \sum_{\substack{l=1 \\ l \neq i}}^{\bar{n}} M_{li} \quad (3.29)$$

3.5 Conclusion and Future Work

A sufficient condition for structural stability is established for QP systems with a positive steady-state point by transforming it to a linear CRN. The resulting conditions are simple inequalities (3.28) and (3.28) that represent sign conditions of the LV parameter vector Λ and matrix M , and the dominant main diagonal property of the latter.

Future work includes the use of time-rescaling to enlarge the possibility of the LV parameters Λ and M to fulfill the above sufficient conditions, the checking of which will lead to solving an LMI.

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