Stability Analysis and Control of Hybrid Systems

PhD thesis

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1 Motivation and aims

Modelling of physical systems is of vital importance both in theoretical research areas and in industrial applications. System models are usually developed for either to understand (analyse) or control (synthesise) them. The ones used in industry are often based on, or originated from first engineering principles [1].

Models of physical systems can be described using various ways including linear or nonlinear approaches. The number of nonlinear representations is countless and it is virtually impossible to find a complete and elegant theory which exists already for the linear case. Presently it is practically hopeless to find universal analytical techniques which are applicable to any nonlinear system with arbitrary input signal. Among those ways, however, one of the most important ways of description is the representation by ordinary differential equations.

Lot of applications demand, in addition, the ability of expressing a sort of discrete switching behaviour: while in nature nonlinearity is the rule rather than the exception, the same is true for being continuous-discrete hybrid system among industrial applications. Appearance of hybrid systems ranges from nuclear reactors [2], to automotive applications [3, 4] or even in robot walking [5]. In literature and practice there can be found techniques to cope with systems showing nonlinearities (LQ, MPC) but those are restricted to systems which show no hybrid behaviour. This need of handling nonlinear hybrid systems motivated by their usefulness in practice triggered the present dissertation.

One important task is to control hybrid systems. A classical approach for this is to solve the corresponding generalized quasi-variational inequalities (generalized forms of hybrid Bellmann-equations) by adding costs accordingly to the jumps and changes of states [6]. There has been notable progress in this area, but the methods to solve them still suffer from the curse of dimensionality.

Another important problem is to check the stability of hybrid systems. There are procedures known both for discrete and continuous systems to decide whether or not a given hybrid system is stable. It is possible to produce Lyapunov functions by concatenating quadratic Lyapunov-like functions [7, 8] or entirely other methods like construct them e.g. using sum of squares [9]. However it is not easily to find the domain of attraction (DOA) of hybrid systems even with the above mentioned Lyapunov functions. In complicated cases often the only method to find the DOA is to examine pointwise a certain environment of the equilibrium point in question. In this dissertation it is planned to give a method to find the DOA of a class of piecewise continuous systems.

In this dissertation an algorithm has been planned to find the domain of attraction for nonlinear hybrid systems which is demonstrated also in several industrially relevant case studies. Another aim of this dissertation is to analyse and control special hybrid systems exploiting their special properties which make it easier to handle the necessary mathematical apparatus.

One aim is—by exploiting certain properties of the systems to be investigated—to design a controller or find the DOA for important systems. The considered nonlinear hybrid system classes are hybrid system of switching type and nonlinear systems with discrete inputs. These properties allow one to handle them analytically, without using heuristics.

In this dissertation important and interesting systems, certain subsystems of the Paks Nuclear Power Plant has been chosen as case studies of analysis and synthesis. In building and analyzing of systems showing hybrid behaviour the controller synthesis and stability analysis often emerge which is a challenging area of system science with many computationally hard problems. Often it is not enough to know if a system (either it is controlled or not) is stable but also important to define its domain of attraction. Therefore, the main focus of the work has been to develop computationally feasible methods for DOA construction and demonstrate them on industrially relevant case studies.

2 Estimation of DOA of nonlinear autonomous systems

In the following we will consider the ordinary autonomous differential system in the form of

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Moreover we assume that for each $x \in \mathbb{R}^n$ a unique solution $\phi(t, x)$ exists which is defined on $\mathbb{R}$ such that $\phi(0, x) = x$. In the followings it is also assumed that the function f is smooth enough that Eq. (1) has a unique solution corresponding to each initial condition $x(0) = x_0$.

The core concepts of the apparatus are given below.

Definition 1. The *domain of attraction* of the origin is the set

$$S(0) = \{\, x_0 : x(t, x_0) \to 0 \text{ as } t \to \infty \,\} \tag{2}$$

where $x(t, x_0)$ denotes the solution of the system in Eq. (1) corresponding to the initial condition $x(0) = x_0$.

Definition 2. A scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is called *Lyapunov function* for the system (1) if it is locally positive definite about the origin (i.e. on some neighbourhood $\mathcal{U}$ about the origin it is true that $V(x) > 0$, $x \in \mathcal{U} \setminus \{0\}$ and $V(0) = 0$). If moreover the time derivative of V is locally negative semi-definite about the origin (i.e. on some neighbourhood $\mathcal{V}$ about the origin it is true that $\dot{V}(x) \leq 0$, $x \in \mathcal{V} \setminus \{0\}$) then the origin is proven to be stable.

It is well known that even if a Lyapunov function exists to an autonomous ODE, then it is not unique. A maximal Lyapunov function is a special Lyapunov function on A which indicates the DOA for a given locally asymptotically stable equilibrium point.

Definition 3. A function $V_M : \mathbb{R}^n \to \mathbb{R}_0^+$ is called *maximal Lyapunov function* for the system (1) if

- $V_M(0) = 0$, $V_M(x) > 0$, $x \in A \setminus \{0\}$
- $V_M(x) < \infty$ if and only if $x \in A$
- $\dot{V}_M$ is negative definite over A and
- $V_M(x) \to \infty$ as $x \to \partial A$ and/or $|x| \to \infty$,

with A being the domain of attraction of the origin for system (1).

The following statement grounds the algorithm to find the DOA for a non-linear autonomous system.

Proposition 4. *Suppose we can find a set $B \subseteq \mathbb{R}^n$ containing the origin in its interior, a continuously differentiable function $V : B \to \mathbb{R}_+$ and a positive definite function ϕ such that*

1. $V(0) = 0, V(x) > 0 \forall x \in B \setminus \{0\}$
2. $\nabla V(x)' f(x) = -\phi(x) \forall x \in B$
3. $V(x) \to \infty$ *as* $x \to \partial B$ *and/or* $\|x\| \to \infty$.

Then $B = A$.

Suppose that function f can be expressed in Taylor series expansion

$$\dot{x} = f(x) = \sum_{i=1}^{\infty} F_i(x). \tag{3}$$

where functions $F_i, \sim i \geq 1$ are again homogeneous functions of degree i. For $i = 1$ it is $F_1(x) = \Phi x, \sim \Phi \in \mathbb{R}^{n \times n}$, where Φ is the Jacobian matrix of f at $x = 0$. For the sake of brevity, let $F_i(x) = 0,\ i \leq 0$. This assumption makes it possible to express the forthcoming equations in simpler and more concise form, since at some combinations of the summation indices there appear F_is with non-positive is.

Based on the properties of the maximal Lyapunov functions, we seek for a function V_M and a positive definite function ψ satisfying $V_M(0) = 0$ and

$$\dot{V}_M(x) = -\psi(x) \tag{4}$$

over some neighbourhood of the origin such that the set ∂A is given by the relation $V_M(x) \to \infty$.

The candidate Lyapunov function should excess any limit as x gets closer to the boundary of set S or as $\|x\| \to \infty$. If it was possible to write V_M as $V_M(x) = \frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials in x then ∂A would be given by $D(x) = 0$. This constructions follows to a structure which can in short be expressed as an under-determined set of linear equations

$$A_n y = b_n \tag{5}$$

where A_n are matrices of appropriate dimension, and the vector y is composed of the coefficients of the homogeneous functions R_is and Q_is.

Let us find homogeneous functions R_n and $Q_{n-2}, n \geq 3$, such that the coefficients of R_n and Q_n solve the constrained minimization problem yielded by Eq. (5)

$$\begin{array}{c} \min e_n(y) \\ \text{s.t. } A_n(y) = b_n \end{array} \tag{6}$$

where $e_n(y)$ is the squared 2-norm of the coefficients of degree greater than or equal to $n + 1$ in the expression of $\dot{V}_n$.

According to the theorem of LaSalle [10] about invariant sets, one can choose the largest positive value $C^\star$ such that the sublevel set

$$A_{\text{est}} = \{ x : V_n(x) < C^\star \} \tag{7}$$

is contained in the region given by

$$\Omega = \left\{ x : \dot{V}_n(x) \leq 0 \right\}. \tag{8}$$

As soon as the desired accuracy (computed as $e_n(y^\star) = 0$ for the minimizer $y^\star$) has been reached (or the error starts growing) the iteration can be stopped.

Relation $A_{\text{est}} \subseteq A$ holds true in each step which means the more step one executes the more precise estimation of A can be computed.

If $e_n(y^\star) = 0$ for some $y^\star$ and some n then the iteration can be stopped and due to the relation $\dot{V}_n = -x'\mathfrak{Q}x$ the set A is given by $D(x) = 0$, i.e.

$$A = \left\{ x : \sum_{i=1}^{n-2} Q_i(x) > -1 \right\}. \tag{9}$$

The advantage of this algorithm is that one does not have to know the solution of the system starting from different initial values; only a minimization problem (a linear programming problem) needs to be solved in each step of the iterative approximation procedure. Moreover, the applicable system class is wider than that of the majority of available algorithms can handle (they are mainly restricted to polynomial systems).

In case of the steam generator it can be seen that the proposed algorithm gives a usable subset of the scanned region but its calculation time is just a fraction of the latter. However it was not able to reproduce the 'tail' of the DOA that is seen in the scanned region and also suggests some points to be stable which are not in real. Still, it can be said that the stability results provided by the algorithm are sufficient enough to investigate the dynamics of the two operating modes of the power controller.

3 The DOA of hybrid nonlinear systems

There are two possible fundamentally different ways of describing a hybrid system that possesses both continuous and discrete variables. The first one utilizes the concepts of theory of discrete event systems (see [11], [12]) and embeds the continuous elements into a discrete system model.

In this dissertation we follow the other approach which sometimes is called *switched systems approach* i.e. some non-smooth variables are embedded into the continuous part so that we describe the dynamics by piecewise functions which are continuous, however, not necessarily differentiable on the border between different dynamics.

To find the common DOA $A (= A(\{0\}))$ using the Vanelli-Vidyasagar algorithm, the following criteria should be fulfilled for f:

1. f is continuous,
2. $f_i(0) = 0, \forall i \in \bar{n}$,
3. $0 \in \mathcal{I}(\cap_{i\in\bar{n}} A_i)$,
4. f should be Lipschitz continuous, which due to Lagrange-theorem, holds if f_i is Lipschitz-continuous on X_i for each $i \in \bar{n}$, and

5. in order to directly use the implemented algorithm in *Mathematica*, f should be differentiable on some neighbourhood about the origin. Note that on set A_i the function V_i is maximal Lyapunov function for system f_i.

It is seen from conditions (1)-(5) that if f is continuously differentiable about the origin or the origin lies within $\mathcal{I} \cap_{i\in\bar{n}} X_i$ then the following algorithm can be used on the hybrid system determined by f.

The method constructs an overall Lyapunov function from the individual Lyapunov functions of the sub-dynamics in a rational function form by approximating a maximal Lyapunov function V_i. In addition, a potentially conservative but computationally feasible method is proposed to estimate the overall DOA from the individual sub-Lyapunov functions using a maximal fitting hypersphere. A manually tuned approximation method based on the manual tuning of the level value has also been proposed for two-dimensional systems. The performance of the method was investigated by simulations using a two-dimensional example system.

Following another approach, the DOA of an existing industrial plant of great importance, the primary circuit of a nuclear power plant has been investigated. The method used to determine the DOA is based on purely analytical considerations. A two dimensional autonomous hybrid differential equation model of the controlled primary circuit has been developed for the purpose of the stability analysis. It has been shown that the DOA of the controlled system encapsulates the operating domain for any positive value of the considered feedback gain. It was also shown that the gain has a great impact of the dynamics of the controlled system and a practically advantageous domain of the gains has been determined (see [13]).

4 Application of multiparametric programming to controller design for discrete input hybrid nonlinear systems

The investigated system belongs to a special subclass of hybrid systems, the class of so-called input-discrete hybrid systems. These systems have continuous states but their inputs are discrete due to technological reasons.

4.1 System description

The state-feedback controller designed for the discretized form comes as a set of polyhedra and a PWA control-law given in the form (if no feedback prestabilization is enabled)

$$U(k) = F_i^r x(k) + G_i^r \tag{10}$$

where r denotes the active polytopic region from the set $\{\mathcal{P}_r\}_{r=1}^{R}$ in which the controller actually is. The dynamics which is active in a particular partition is denoted by i which is constant 1 in our case and the cost associated to the state $x(k)$ is calculated by the formula

$$J = x(k)^T \mathcal{A}_i^r x(k) + \mathcal{B}_i^r x(k) + \mathcal{C}_i^r \tag{11}$$

Matrices F_i^r, G_i^r, $\mathcal{A}_i^r$, $\mathcal{B}_i^r$, $\mathcal{C}_i^r$ and set of polytopes $\{\,\mathcal{P}_r\,\}_{r=1}^{R}$ are yielded as the result of solving the CFTOC. The control algorithm runs over a polyhedral partition (since the cost function is a piece-wise affine function over a set of polyhedra) of possible states and it ensures the stability of the system.

First it has to be identified which is the active region the system is in. If there are overlapping regions then the cost is calculated for each region—since the cost is defined differently for each of them (see Eq. (11))—and choose the region for which it is minimal. Once the active region is identified then the control action can be obtained by Eq. (10).

After solving the optimal control problem for a fixed prediction horizon N, evaluation of Eq. (10) gives a vector of control moves which minimizes the given performance criterion. When applying the obtained control law in the closed-loop, only the first input $u\,(0)$ is extracted from the sequence and is applied on the system. This policy—of which procedure is iterated during the control process—is referenced to as the Receding Horizon Policy (see [14]).

5 New scientific results

The new scientific results presented are summarized in the following Theses.

Thesis 1. Estimation of DOA of nonlinear autonomous systems (Chapter 3) (**[R2]**,**[R3]**,**[R4]**)
An improved algorithm based on constructing maximal Lyapunov functions [15] is proposed to estimate the DOA of nonlinear autonomous systems. The advantage of this algorithm is that one does not have to know the solution of the system starting from different initial values; only a minimization problem (a linear programming problem) needs to be solved in each step of the iterative approximation procedure.

- The proposed algorithm can be applied on systems of any dimension. However, due to the automatic calculation of $C^{\star}$ in Eq. (7), the DOA will reside inside an appropriately chosen sphere-environment of the origin, which can be considered to be a conservative approximation of the widest sublevel set.
- The proposed algorithm has been used for an industrially relevant case-study. Two subsystems of the primary circuit in the Paks Nuclear Power Plant was investigated, one is the reactor characterized by neutron flux (N) and the other one is the steam generator. It has been found that the DOA estimated by the proposed algorithm is almost exactly matches the real DOA.

Thesis 2. Estimation of DOA of hybrid nonlinear systems (Chapter 4) (**[R5]**,**[R6]**,**[R7]**)
Two different methods have been proposed to estimate the DOA of two different types of switching hybrid systems that use the Lyapunov functions of the individual dynamics and construct the DOA therefrom.

- The first method is based on the algorithm proposed in Chapter 3, and it is capable to give a subset of the DOA for non-linear hybrid (switching) systems where the dynamics is continuous on the boundary of the different regimes of the state space. The method constructs an overall Lyapunov function from the individual Lyapunov functions of the sub-dynamics in a rational function form by approximating a maximal Lyapunov function. All necessary functions and algorithms have been implemented in a *Mathematica*-package.
- As an industrial relevant case study the DOA of an existing industrial plant of great importance, the primary circuit of a nuclear power plant has been investigated. The method used to determine the DOA is based on purely analytical considerations. A two dimensional autonomous hybrid differential equation model of the controlled primary circuit has been developed for the purpose of the stability analysis containing the temperatures of the liquid in the primary circuit and of the water in the pressurizer. It has been shown that the DOA of the controlled system encapsulates the operating domain for any positive value of the considered feedback gain. It was also shown that the gain has a great impact of the dynamics of the controlled

system and a practically advantageous domain of the gains has been determined.

Thesis 3. Application of multiparametric programming to controller design for discrete input hybrid nonlinear systems (Chapter 5)
([**R4**])
In this thesis a PWA state-space model with discrete-valued inputs has been developed for controller design purposes by solving the corresponding CFTOC problem. The effect of parameters of the model and also of the controller has been investigated on the quality of the controller performance and computational properties.

- As an industrial relevant case study a simple two dimensional model has been developed for the pressurizer of the nuclear reactor. This model has discrete inputs which makes it to fall to the hybrid model class. A controller for this model was designed by solving the CFTOC-problem using multi-parametric programming techniques.
- It can be ascertained that the prediction horizon and the sampling time are the two most important tuning parameters. To get shorter calculation times and to stabilize the closed-loop system it is highly encouraged to choose the longest sampling time and the shortest prediction horizon possible. The best controller-parameters have been chosen by performing preliminary simulation experiments. It was found to be irrelevant whether 1-norm or infinity-norm is chosen.

6 Publications

The results of this thesis were previously either published in a journal or in form of research report or presented in conferences as enlisted below.

[R1] Sz. Rozgonyi, K. M. Hangos, Hybrid modelling and control of an industrial vaporizer, Proceedings of the 15th International Conference on Process Control, Slovakia, 2005.

[R2] Sz. Rozgonyi, K. M. Hangos, Improved estimation method of region of stability for nonlinear autonomous systems, 7th International PhD Workshop, Czech Republic, 2006.

[R3] Sz. Rozgonyi, K. M. Hangos, G. Szederkényi, Estimating the stability region of a controlled pressurized water reactor, 8th International Conference on The Modern Information Technology in the Innovation Processes of the Industrial Enterprises, Budapest, 2006.

[R4] Sz. Rozgonyi, K. M. Hangos, G. Szederkényi, Improved estimation method of region of stability for nonlinear autonomous systems, Tech. rep., Systems and Control Laboratory, Computer and Automation Research Institute, http://daedalus.scl.sztaki.hu/pdf/research reports/SCL-002-2006.pdf, 2006.

[R5] Sz. Rozgonyi, K. M. Hangos, Estimating the region of stability for a hybrid model, Proceedings of the 17th International Conference on Process Control, Slovakia, 2007.

[R6] Sz. Rozgonyi, K. M. Hangos, G. Szederkényi, Determining the domain of attraction of hybrid non-linear systems using maximal Lyapunov functions, Kybernetika 46 (1) (2010) 19–37.

[R7] Sz. Rozgonyi, K. M Hangos, Domain of attraction analysis of a controlled hybrid reactor model, Annals of Nuclear Energy 38 (5) (2011) 969–975.

7 Future work

There are further development directions regarding the results of the dissertation.

- The estimated DOA based on the maximal fitting hypersphere could be substantially improved by either introducing correct heuristics or finding properly the value of $C^\star$ in Eq. (7) corresponding to a not spherical environment of the asymptotical stable equilibrium point achieving less conservative estimations of the DOA.
- The model of the primary circuit described in section 4.2.1 could be improved by including the hysteresis of the pressure controller.
- The visualization capabilities of the implemented Mathematica-package could further be improved by introducing higher dimensional slices of the maximal Lyapunov-function and its derivative. This could lead further investigation of possible heuristics on choosing appropriate $C^\star$ values.

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